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Discrete Optimization for Shape Matching

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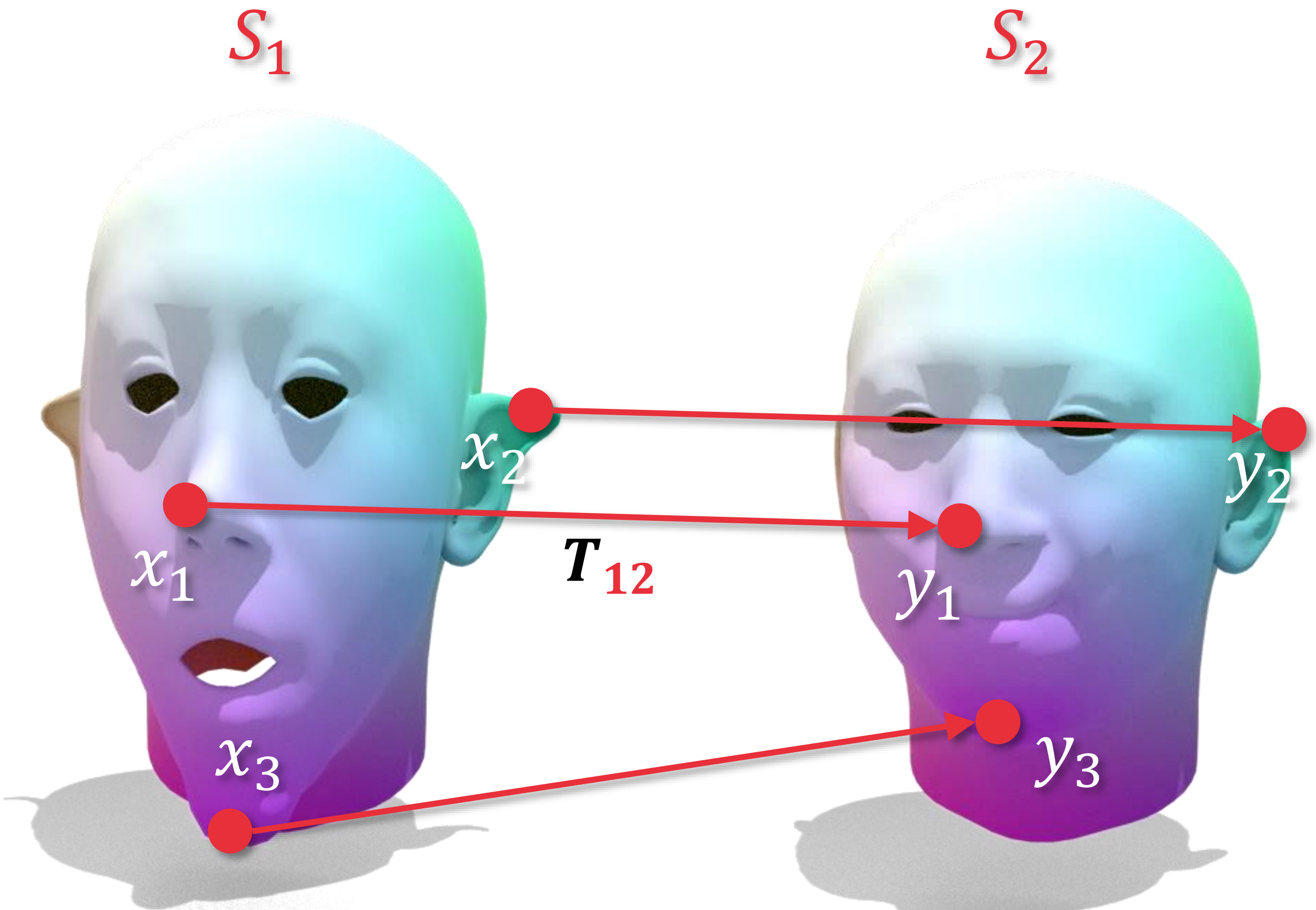
Introduction to Shape Matching



Notation: $\boldsymbol{T}_{12}, \boldsymbol{C}_{12}, \boldsymbol{\mathcal{P}}_{12}: \boldsymbol{S}_1 \rightarrow \boldsymbol{S}_2$

Point-based methods

- [Bronstein et al. 2006],
- [Huang et. Al 2008]...

Parameterization-based methods

- [Lipman and Funkhouser 2009]
- [Aigerman et al. 2017]...

Optimal transport

- [Solomon et al. 2016]
- [Mandad et al. 2017]...

Functional maps

- [Ovsjanikov et al. 2012]
- [Ezuz and Ben-Chen 2017]...

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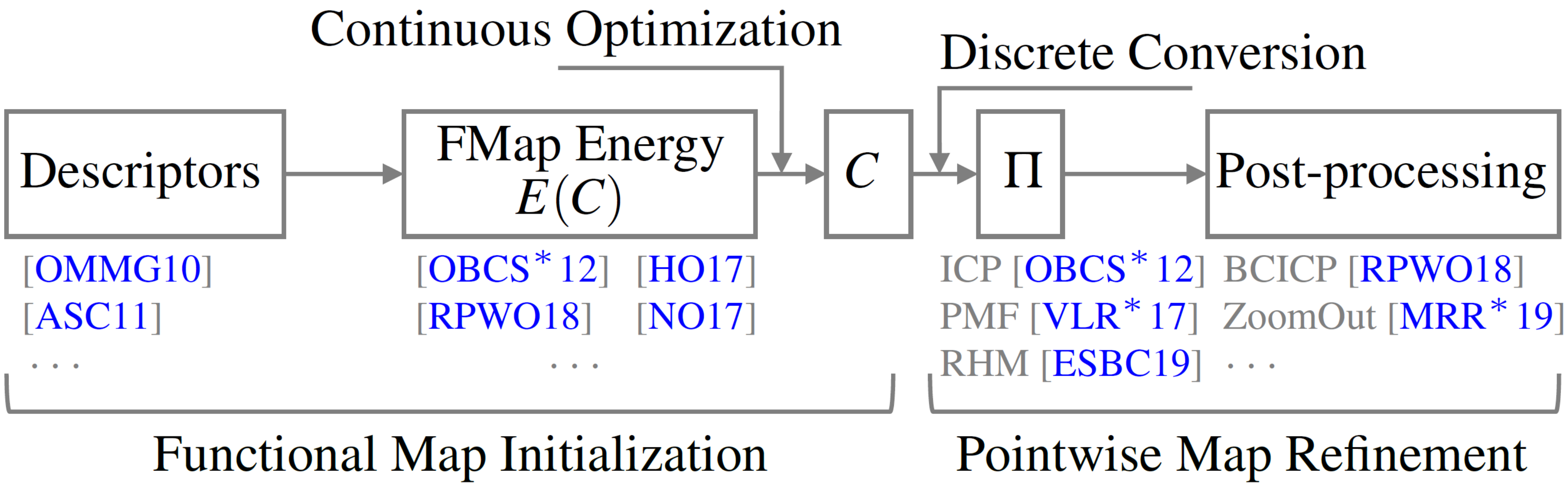
Standard Functional Map Pipeline



Observations:

- different energies for fMap optimization and post-processing
- optimize fMap with regularizers but no hard constraints: $\min_C E(C)$
  - For some $E(C)$ such as Laplacian Commutativity, the global minimizer is zero matrix

Outline

1. Introduction to shape matching
2. Standard functional map pipeline for shape matching
3. Proper functional map
4. Discrete solver for functional map optimization
5. Results: evaluations & applications

Proper Functional Map Space

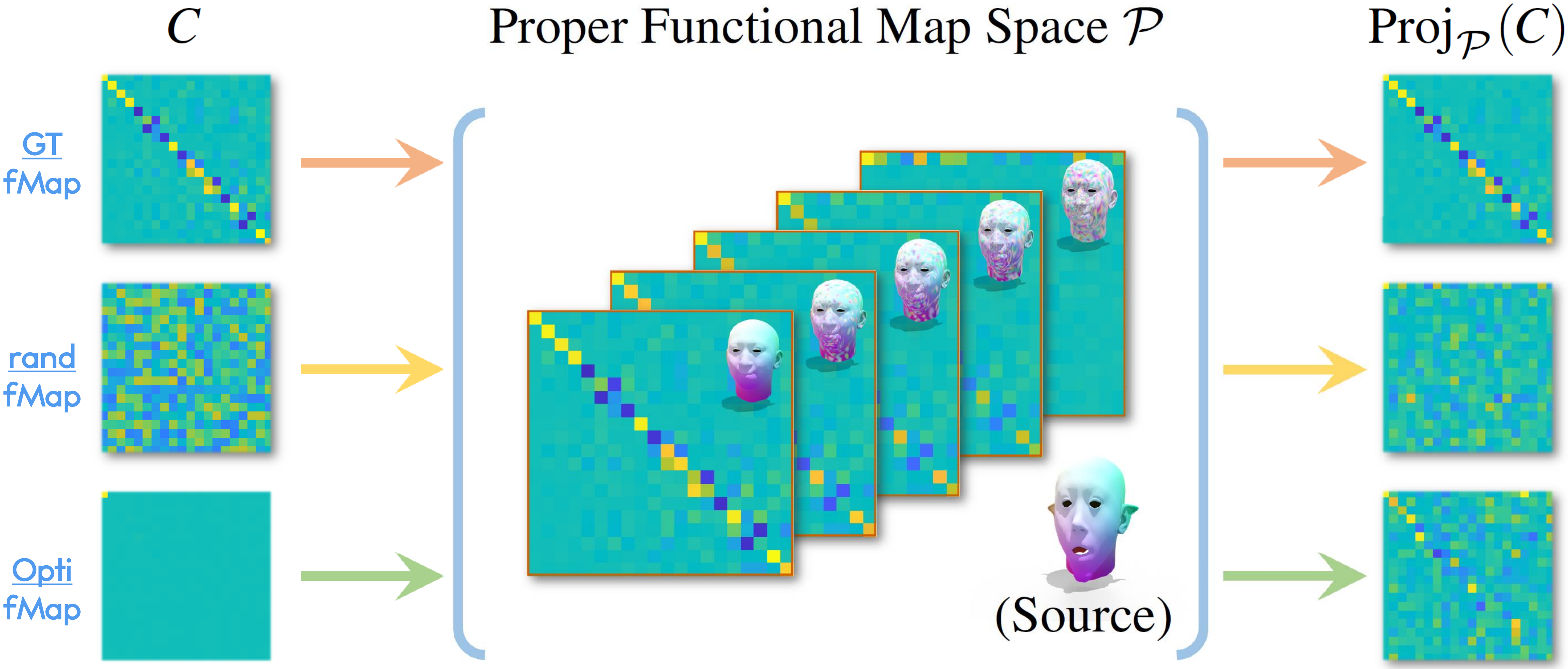
Definition: The proper functional map space is the set of functional maps that arise from pointwise correspondences. Particularly, we call a functional map $\boldsymbol{C}_{12}$ proper if there exists a pointwise map $\boldsymbol{\Pi}_{21}$ such that $\boldsymbol{C}_{12} = \boldsymbol{\Phi}_2^{\dagger}\boldsymbol{\Pi}_{21}\boldsymbol{\Phi}_1$

- $\boldsymbol{\Phi}_i$: Laplace-Beltrami eigenbasis of shape $\boldsymbol{S}_i$
- $\boldsymbol{A}^{\dagger}$: Moore–Penrose pseudoinverse of matrix $\boldsymbol{A}$
- $\boldsymbol{\Pi}_{21}$: matrix representation of a pointwise map from $\boldsymbol{S}_2$ to $\boldsymbol{S}_1$, i.e.,
    - if $\boldsymbol{\Pi}_{21}(\boldsymbol{i},\boldsymbol{j}) = \mathbf{1}$, $\boldsymbol{v}_i \in \boldsymbol{S}_2$ is corresponding to $\boldsymbol{v}_j \in \boldsymbol{S}_1$

Notation: The space $\mathcal{P}_{12}$ of proper functional maps between $\boldsymbol{S}_1$ and $\boldsymbol{S}_2$ is denoted as:

$$\mathcal{P}_{12} = \left\{\boldsymbol{C}_{12} \middle| \ \exists \boldsymbol{\Pi}_{21}, \text{s.t.} \quad \boldsymbol{C}_{12} = \boldsymbol{\Phi}_2^{\dagger}\boldsymbol{\Pi}_{21}\boldsymbol{\Phi}_1\right\}$$

Proper Functional Map Space



Proper Functional Map Space

Previous Formulation:

$$\min_{C_{12}} E(C_{12})$$

- Pros: easy to solve
- Cons: not proper, i.e., converting to a pMap can introduce errors

Our Formulation:

$$\min_{C_{12} \in \mathcal{P}_{12}} E(C_{12})$$

- Pros: returns a proper fMap
- Cons: search space is discrete and exponential in size

[NO17]:

$$\min_{C_{12}} E(C_{12}) + E_{\text{multiplicative}}(C_{12})$$

Propose the multiplicative operators to guide the fmap to be proper <u>implicitly</u>

Discrete Solver for Functional Map Optimization

Problem Formulation:

$$\min_{C_{12} \in \mathcal{P}_{12}} E(C_{12})$$

Naïve solution 1:

1. $C^* = \mathrm{argmin}\, E(C_{12})$
2. $C = \mathbf{Proj}_{\mathcal{P}_{12}}(C^*)$

- Pros: easy to solve
- Cons: the unconstrained optimization can lead to undesirable local minima; the projection step is discrete and can introduce errors

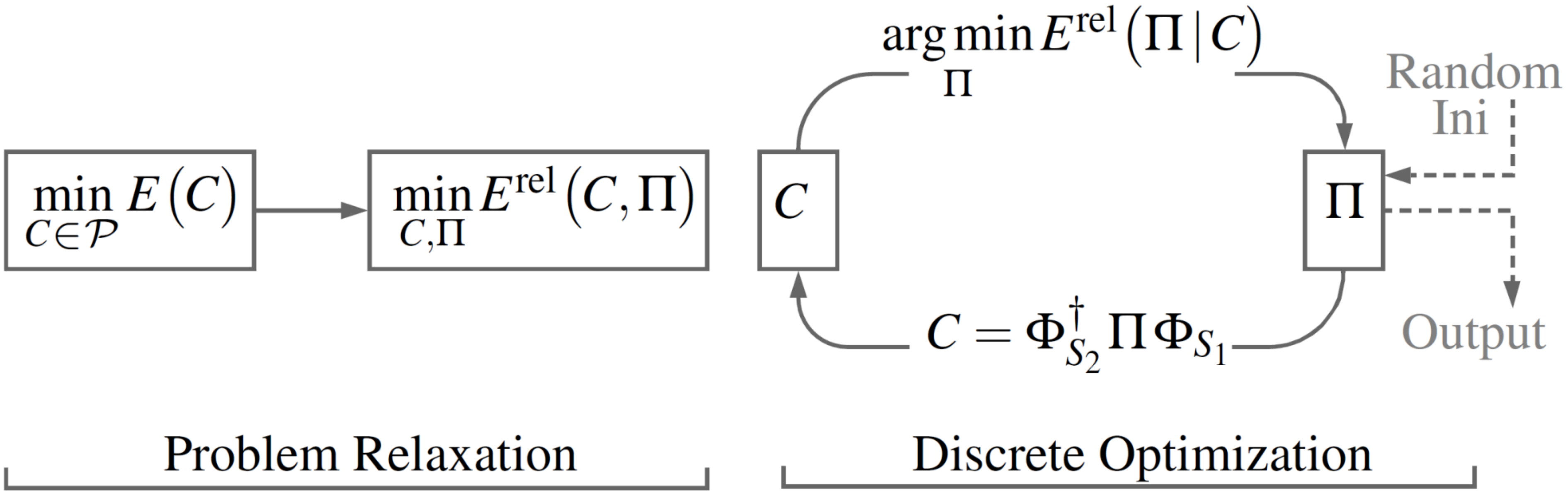
Naïve solution 2: $\mathcal{P}_{12}$ is discrete, and we can enumerate all possible proper functional maps to find the global minimizer

- Pros: returns global minimizer
- Cons: $\mathcal{P}_{12}$ is toooooo large, only works for shapes with less than 10 vertices

Discrete Solver for Functional Map Optimization

Approach Overview:

1. Given a functional map energy, reformulate it by replacing some terms $\boldsymbol{C}_{12}$ with $\boldsymbol{\Phi}_2^{\dagger}\boldsymbol{\Pi}_{21}\boldsymbol{\Phi}_1$.
2. Add a coupling term to the energy and make the functional map $\boldsymbol{C}_{12}$ and pointwise map $\boldsymbol{\Pi}_{21}$ independent free variables of the resulting problem.
3. Alternate between computing the optimal functional and pointwise maps, while fixing the other representation. (ZoomOut and sampling techniques can be applied here)



Discrete Solver for Functional Map Optimization

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Lemma 4.1: Given arbitrary matrices X, Y, and a reduced basis Φ, s.t. $\Phi^T A \Phi = Id$, then the following two problems:

i. $\min_{\Pi} \| \Phi^{\dagger}\Pi X - Y \|_F^2 + \left\| \left(Id - \Phi\Phi^{\dagger} \right) \Pi X \right\|_A^2$, where $\|W\|_A^2 = \mathrm{tr}(W^T A W)$

ii. $\min_{\Pi} \| \Pi X - \Phi Y \|_F^2$

are equivalent. Moreover problem ii) is row-separable and can be solved in closed form through nearest neighbor search. ([EBC17] provides a special case of this statement.)

- [EBC17] EZUZ D., BEN-CHEN M.: Deblurring and denoising of maps between shapes. Computer Graphics Forum 36, 5 (2017)

Discrete Solver for Functional Map Optimization

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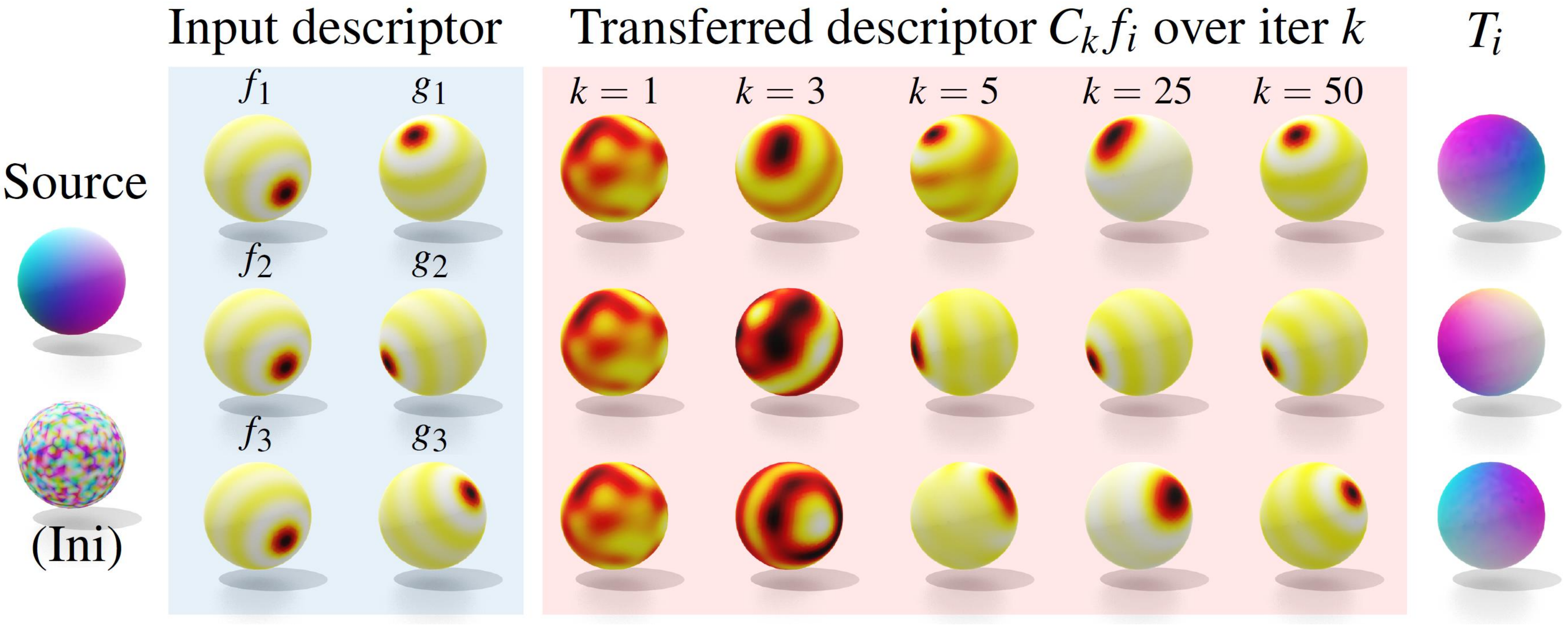
Example of minimizing $\boldsymbol{E}(\boldsymbol{C}_{12}) = \|\boldsymbol{C}_{12}\boldsymbol{f}_1 - \boldsymbol{f}_2\|_F^2$:

1. Apply reformulation: $\boldsymbol{E}^{\mathrm{mod}}(\boldsymbol{C}_{12}, \boldsymbol{\Pi}_{21}) = \left\|\boldsymbol{\Phi}_2^{\dagger}\boldsymbol{\Pi}_{21}\boldsymbol{\Phi}_1\boldsymbol{f}_1 - \boldsymbol{f}_2\right\|_F^2$
2. Add a coupling term: $\boldsymbol{E}^{\mathrm{rel}}(\boldsymbol{C}_{12}, \boldsymbol{\Pi}_{21}) = \boldsymbol{E}^{\mathrm{mod}}(\boldsymbol{C}_{12}, \boldsymbol{\Pi}_{21}) + \boldsymbol{\alpha}\left\|\boldsymbol{\Phi}_2^{\dagger}\boldsymbol{\Pi}_{21}\boldsymbol{\Phi}_1\boldsymbol{C}_{12}^T - \boldsymbol{I}\right\|_F^2$
3. Optimize in an alternating scheme:
   - $\boldsymbol{\Pi}_{21} = \mathrm{argmin}_{\boldsymbol{\Pi}_{21}}\boldsymbol{E}^{\mathrm{rel}}(\boldsymbol{C}_{12}, \boldsymbol{\Pi}_{21})$
   - $\boldsymbol{C}_{12} = \boldsymbol{\Phi}_2^{\dagger}\boldsymbol{\Pi}_{21}\boldsymbol{\Phi}_1$

Example: Minimize Descriptor-preserving Energy

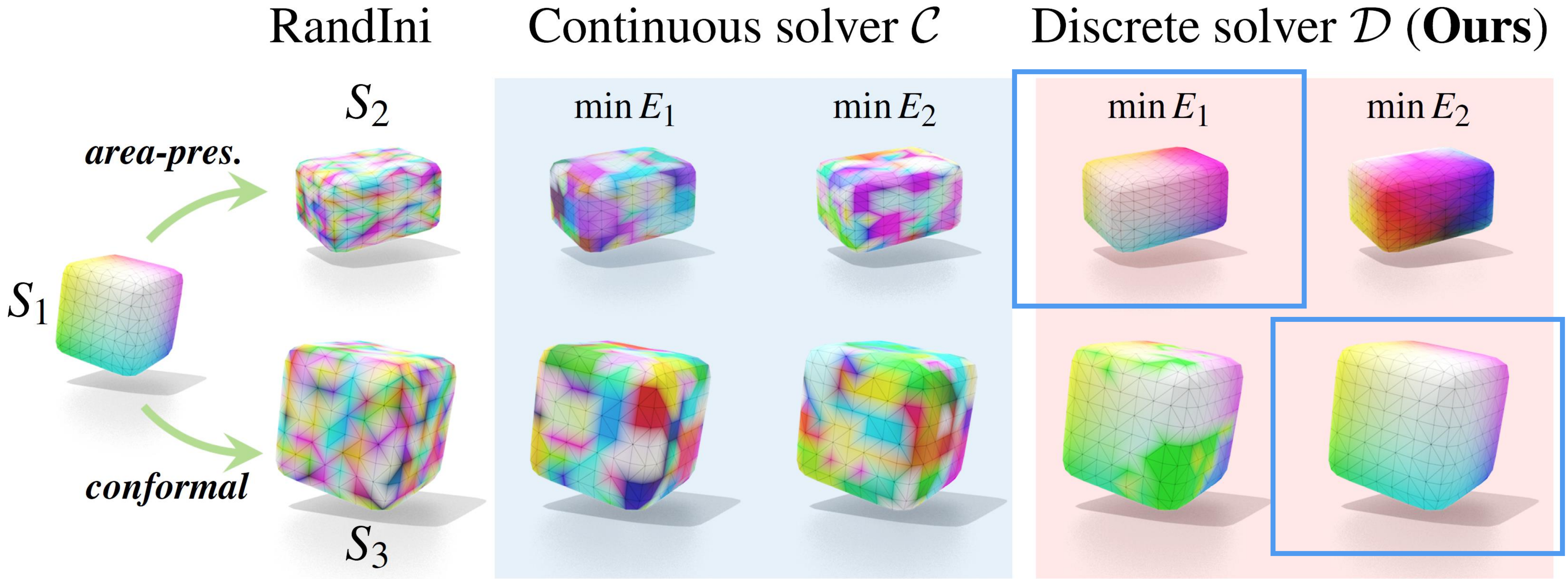
$$\min_{C_{12}\in\mathcal{P}_{12}} \|C_{12}f_i - g_i\|_F^2$$

- (f_i, g_i): given corresponding descriptors



Example: Minimize Area-pres. & Conformal Energy

$$\min_{C_{12}\in\mathcal{P}_{12}} \left\|C_{12}C_{12}^T - I\right\|_F^2 \quad \text{v.s.} \quad \min_{C_{12}\in\mathcal{P}_{12}} \left\|C_{12}\Delta_1 C_{12}^T - \Delta_2\right\|_F^2$$



Contributions

We propose a discrete solver that can optimize functional map-based energies.

- Easy to use/adapt for different energies
- Returns a proper functional map that corresponds to a pointwise map
- Achieves lower energy values compared to the standard continuous solver

Two practical applications:

- **Alternative of Multiplicative Operators**
  - [NO17] proposes the multiplicative operators to guide the optimization towards proper functional maps (implicitly). Our discrete solver outperforms the multiplicative Op.
- **New refinement method: Effective Functional Map Refinement**
  - Combines commonly used fMap energies including bijectivity, orthogonality and Laplacian commutativity from both directions.
  - Achieves better accuracy/bijectivity on SHREC'19

Results: Evaluation on the Discrete Solver

50 Shape pairs from the SMAL dataset

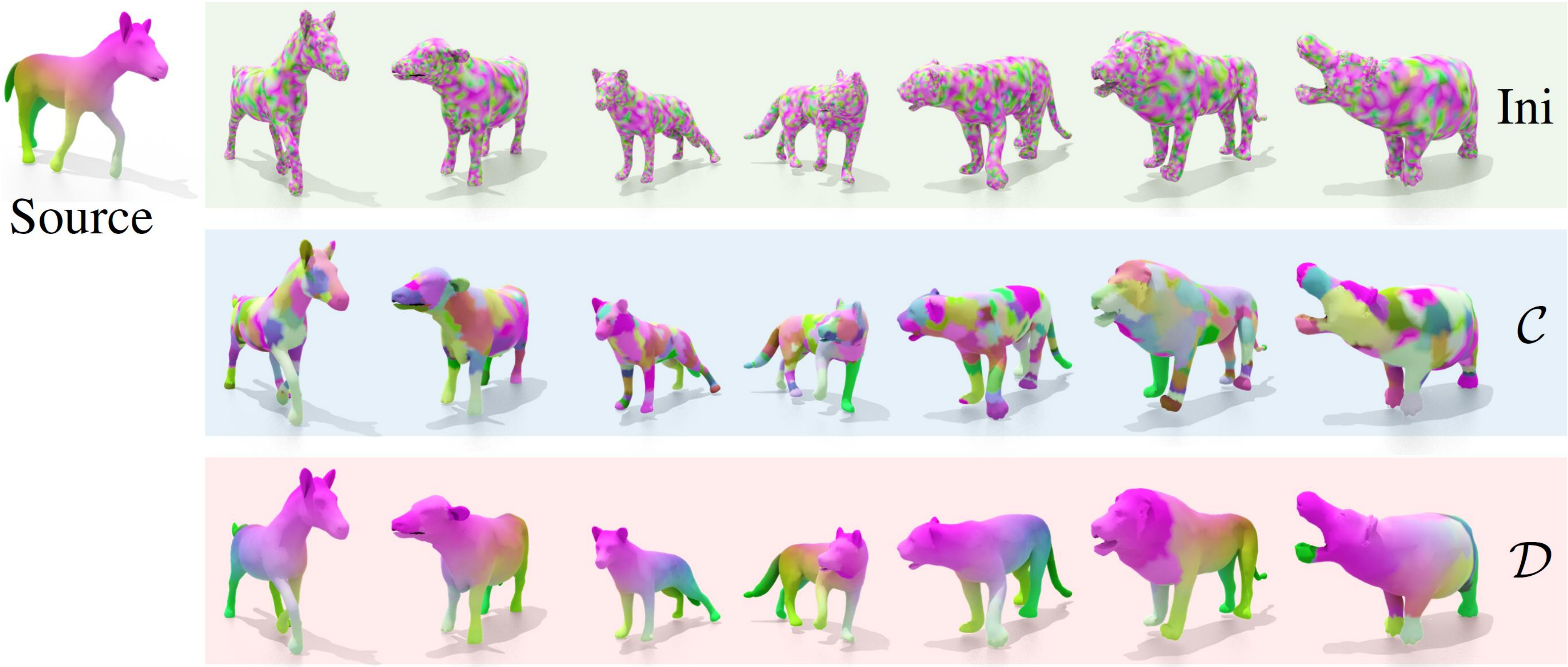
For different functional map-based energies, we compare:

- continuous solver $\mathcal{C}$
- discrete solver $\mathcal{D}$

Energies \ Stats.		min.	avg.	max.	std.
$E_1 = \left\|CC^T - I\right\|_F$	$\mathcal{C}$	4.9924	5.1932	5.4530	0.0976
	$\mathcal{D}$	0.6789	**2.2557**	3.0707	0.4801
$E_2 = \left\|C\Delta_1 - \Delta_2 C\right\|_F$	$\mathcal{C}$	1.0615	1.2261	1.4205	0.0900
	$\mathcal{D}$	0.1575	**0.8053**	1.2197	0.2229
$E_3 = \left\|C\Delta_1 C^T - \Delta_2\right\|_F$	$\mathcal{C}$	1.9219	2.1302	2.2884	0.1005
	$\mathcal{D}$	0.2972	**1.3173**	1.8223	0.3694
$E_4 = \left\|C_{12}C_{21} - I\right\|_F + \left\|C_{21}C_{12} - I\right\|_F$	$\mathcal{C}$	12.531	13.059	13.948	0.2645
	$\mathcal{D}$	0.9355	**4.1369**	5.3564	1.0138

Results: Evaluation on the Discrete Solver

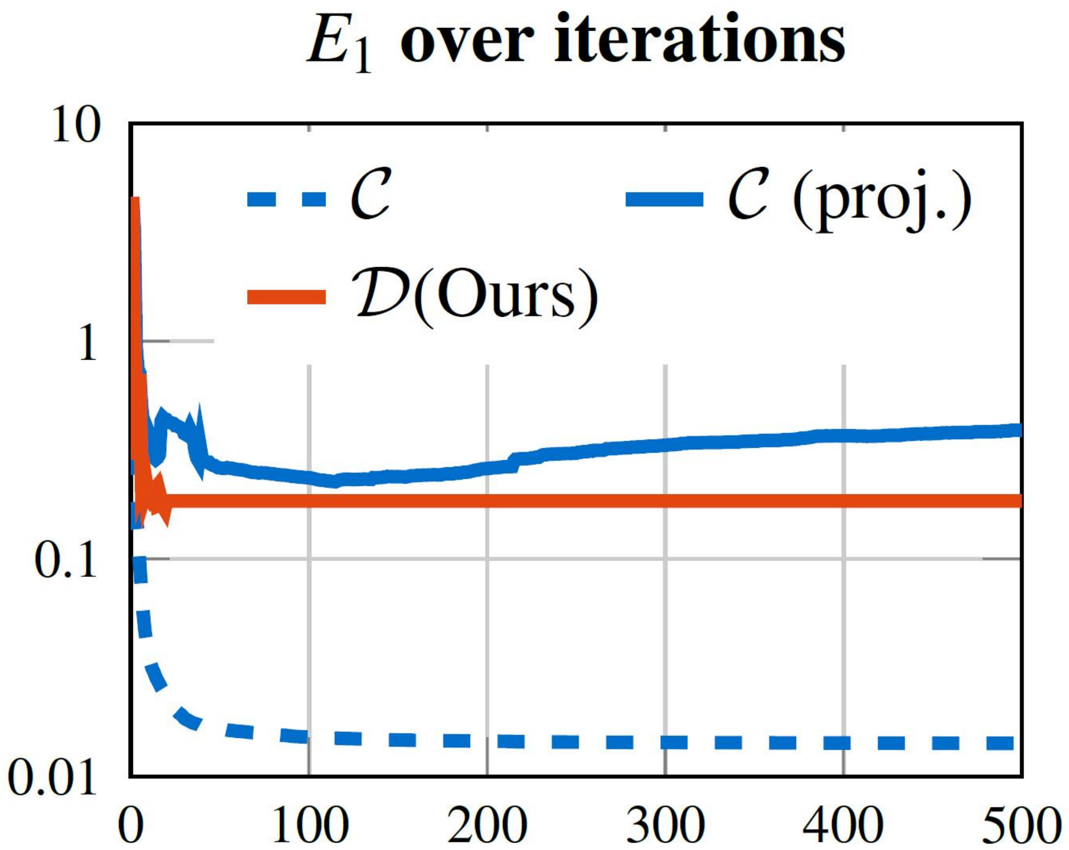
Minimize the Laplacian Commutativity energy

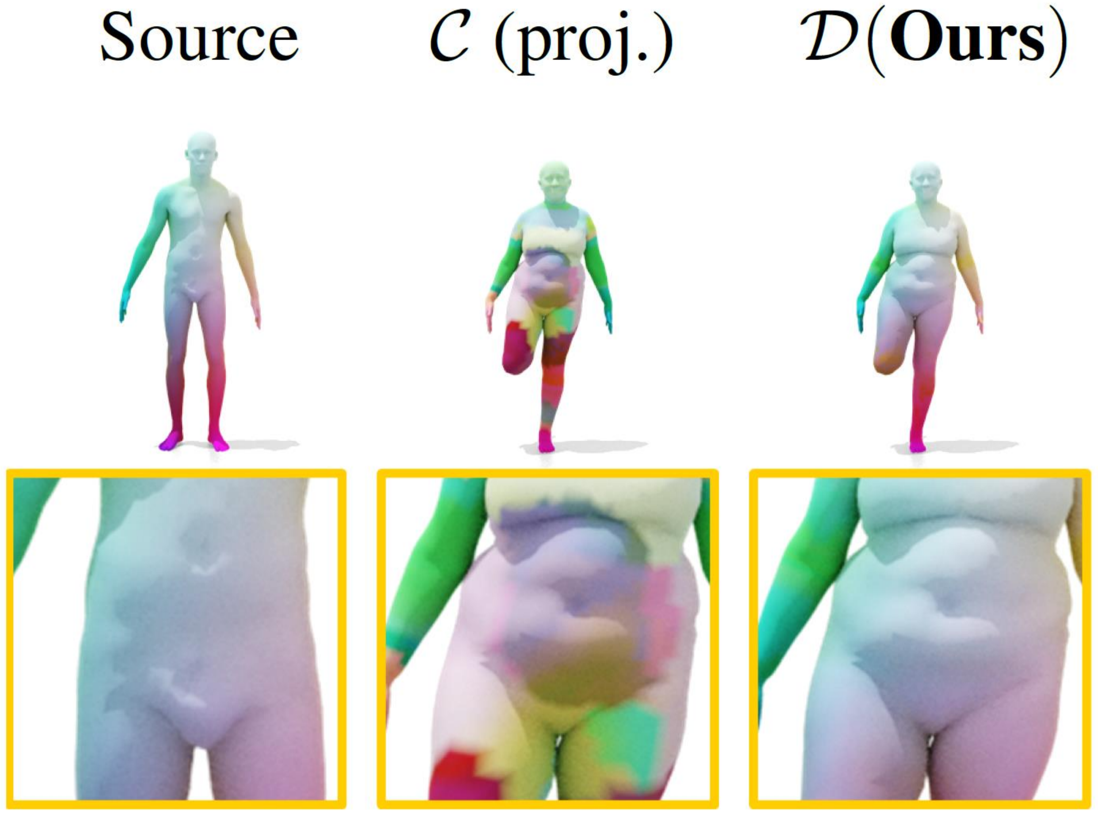


Results: Alternative of Multiplicative Operators

Baseline: $\min_{C_{12}} \sum_i \|C_{12} f_i - g_i\|_F^2 + \alpha \|C_{12}\Delta_1 - \Delta_2 C_{12}\|_F^2 + \beta \sum_i \|C_{12}\Omega_{f_i} - \Omega_{g_i} C_{12}\|_F^2$

Ours: $\min_{C_{12} \in \mathcal{P}_{12}} \sum_i \|C_{12} f_i - g_i\|_F^2 + \alpha \|C_{12}\Delta_1 - \Delta_2 C_{12}\|_F^2$





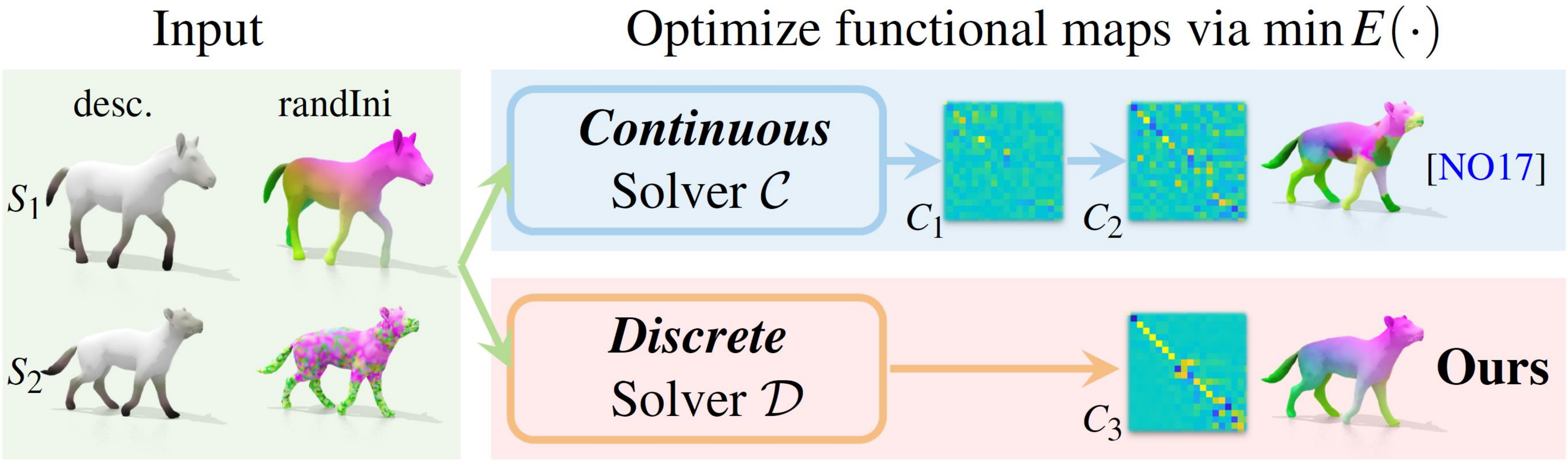
Results: Effective Functional Map Refinement

SHREC'19 Challenge

Methods \ Metrics	Accuracy ($\times 10^{-3}$)	Bijectivity ($\times 10^{-3}$)	Runtime (s)
Initialization	60.4	95.1	-
ICP	47.0	47.4	87.3
PMF (1k)	51.8	**11.8**	118.1
BCICP (5k)	30.1	12.7	437.9
RHM	42.6	13.5	2313
ZoomOut	28.8	26.1	**1.5**
Ours	**27.3**	15.1	8.17

Summary

1. We propose a discrete solver that can optimize functional map-based energies.
2. Two practical applications:
   - Alternative of Multiplicative Operators
   - New refinement method: Effective Functional Map Refinement



Limitations

1. Our discrete solver with the practical modifications has few theoretical guarantees.
2. For some functional map energies with complicated formulations, e.g., using higher order terms, our reformulation strategy might not work directly and more advanced solvers might be needed.

Future Work

1. Explore different coupling terms
2. Investigate different pointwise recovery techniques, such as Sinkhorn algorithm

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Demo code is available at:
https://github.com/llorz/SGP21_discreteOptimization

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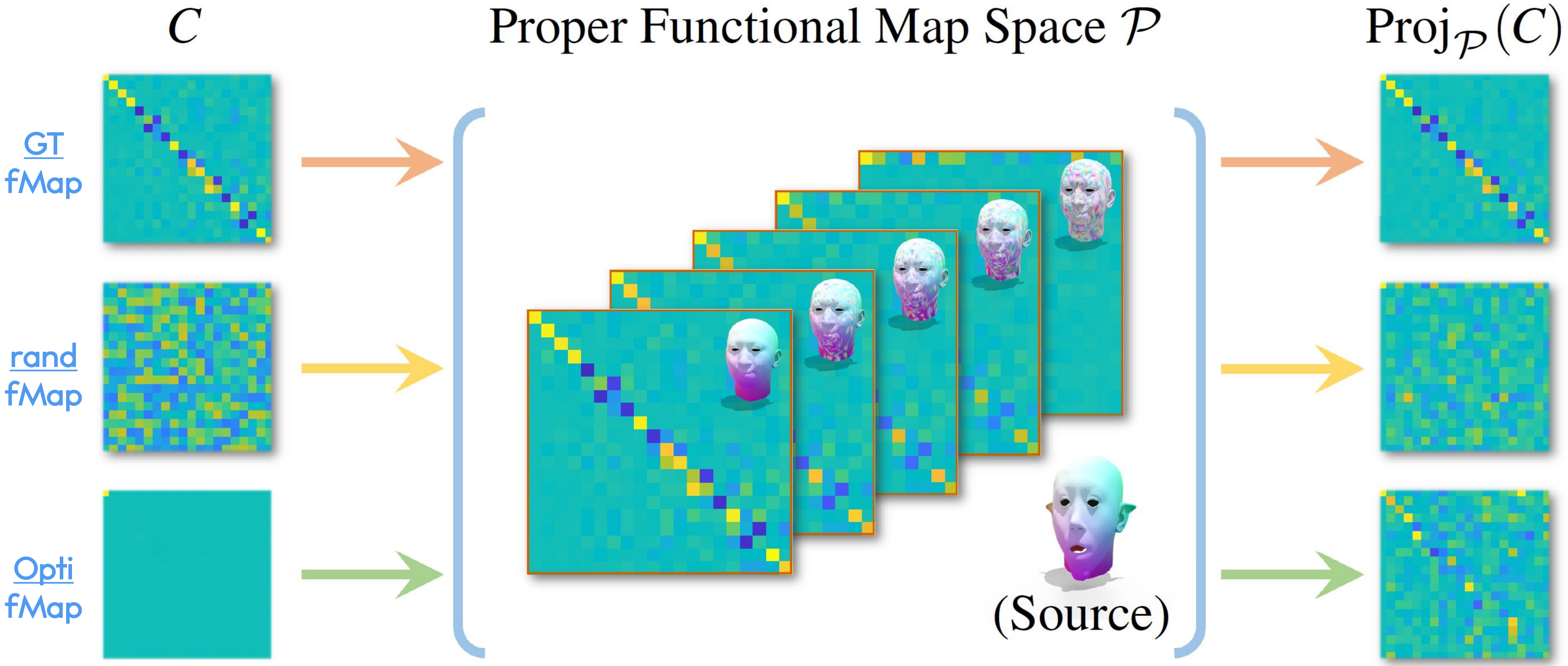
Supplementary

Proper Functional Map Space



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2. Add a coupling term to the energy
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Goal: make it easy to solve $\boldsymbol{\Pi}_{21} = \mathrm{argmin}_{\Pi_{21}} \boldsymbol{E}^{\mathrm{mod}}(\boldsymbol{C}_{12}, \boldsymbol{\Pi}_{21}) + \alpha \left\| \boldsymbol{\Phi}_2^{\dagger}\boldsymbol{\Pi}_{21}\boldsymbol{\Phi}_1 \boldsymbol{C}_{12}^{T} - \boldsymbol{I} \right\|_F^2$

Discrete Solver for Functional Map Optimization

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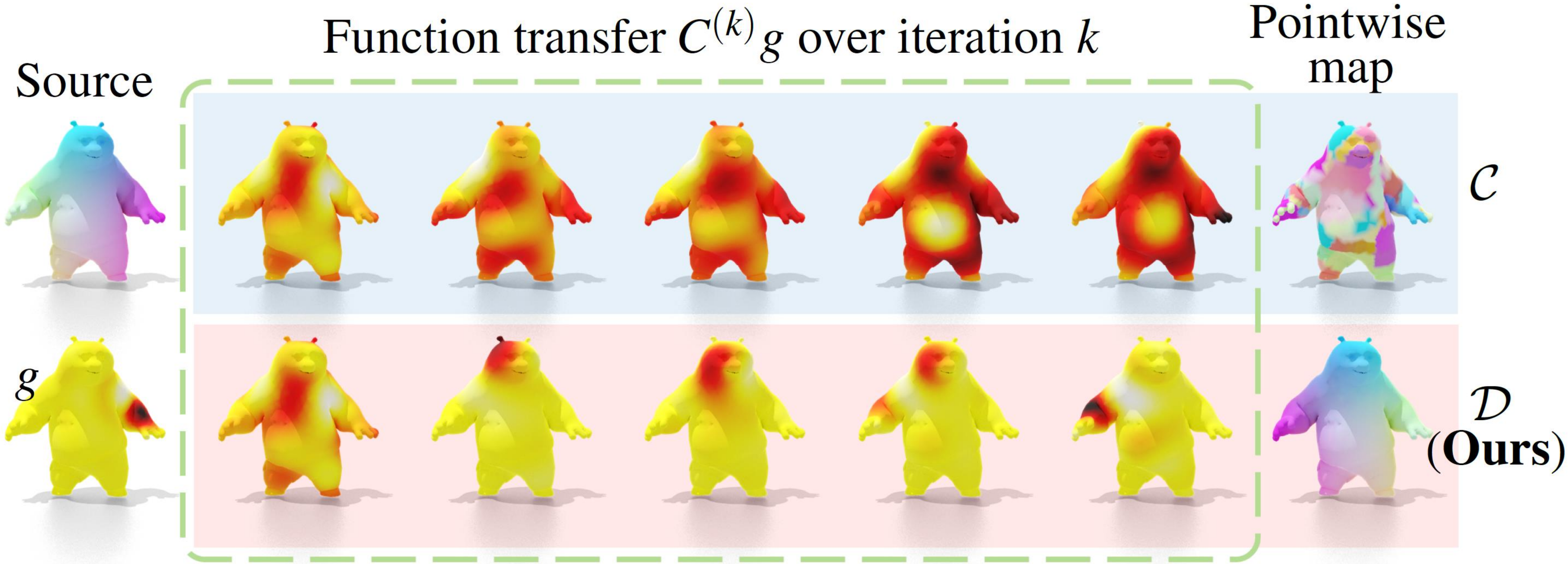
Examples of Reformulation :

- $E(C_{12}) = \|C_{12} f_1 - f_2\|_F^2$ apply reformulation $E^{\mathrm{mod}} = \left\| \Phi_2^{\dagger}\Pi_{21}\Phi_1 f_1 - f_2 \right\|_F^2$
- $E(C_{12}) = \left\| C_{12} C_{12}^T - I \right\|_F^2$ apply reformulation $E^{\mathrm{mod}} = \left\| \Phi_2^{\dagger}\Pi_{21}\Phi_1 C_{12}^T - I \right\|_F^2$
- $E(C_{12}) = \left\| C_{12}\Omega_{f_1} \pm \Omega_{f_2} C_{12} \right\|_F^2$ apply reformulation $E^{\mathrm{mod}} = \left\| \Phi_2^{\dagger}\Pi_{21}\Phi_1 \Omega_{f_1} \pm \Omega_{f_2} C_{12} \right\|_F^2$
- $E(C_{12}) = \left\| C_{12}\Delta_1 C_{12}^T - \Delta_2 \right\|_F^2$ apply reformulation $E^{\mathrm{mod}} = \left\| \Phi_2^{\dagger}\Pi_{21}\Phi_1 \Delta_1 C_{12}^T - \Delta_2 \right\|_F^2$

Example: Minimize Orientation-Reversing Energy

$$\min_{C \in \mathcal{P}} \left\| C\Omega_g + \Omega_g C \right\|_F^2$$

- C: fmap maps to itself
- Ω_g: orientation operator constructed from g



Results: Evaluation on the Discrete Solver

50 Shape pairs from the SMAL dataset

Energies \ ShapeID		1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	Avg. (50 pairs)
$E_1 = \|\|CC^T - I\|\|_F$	$\mathcal{C}$	5.270	5.061	5.304	5.230	5.148	5.213	5.241	5.183	5.041	5.093	5.335	5.122	5.081	5.453	5.287	5.193
	$\mathcal{D}$	2.341	2.573	2.484	2.127	2.477	2.106	2.375	1.821	1.798	2.566	1.264	2.590	2.414	2.386	2.347	2.256
$E_2 = \|\|C\Delta_1 - \Delta_2 C\|\|_F$	$\mathcal{C}$	1.133	1.227	1.156	1.271	1.091	1.123	1.291	1.190	1.125	1.213	1.188	1.113	1.242	1.237	1.170	1.226
	$\mathcal{D}$	0.665	0.465	0.960	1.040	0.818	0.626	0.798	0.626	0.511	0.770	0.158	0.887	0.715	0.894	0.963	0.805
$E_3 = \|\|C\Delta_1 C^T - \Delta_2\|\|_F$	$\mathcal{C}$	2.128	2.172	2.136	2.040	2.010	1.987	1.975	2.197	2.259	2.282	1.997	2.154	2.177	2.275	1.956	2.130
	$\mathcal{D}$	1.412	1.588	1.495	1.193	1.474	1.178	1.729	1.533	1.083	1.451	0.413	1.455	1.568	1.718	0.834	1.317
$E_4 = \|\|C_{12}C_{21} - I\|\|_F$	$\mathcal{C}$	13.02	12.94	13.28	13.16	12.93	13.58	12.93	13.09	13.24	13.16	13.31	12.91	13.95	13.18	13.01	13.06
$+ \|\|C_{21}C_{12} - I\|\|_F$	$\mathcal{D}$	3.895	5.068	4.661	4.926	3.085	4.382	4.074	4.448	4.657	4.716	1.299	4.694	4.380	4.957	4.742	4.137

For different functional map-based energies, we compare:

- standard continuous solver $\mathcal{C}$
- our discrete solver $\mathcal{D}$