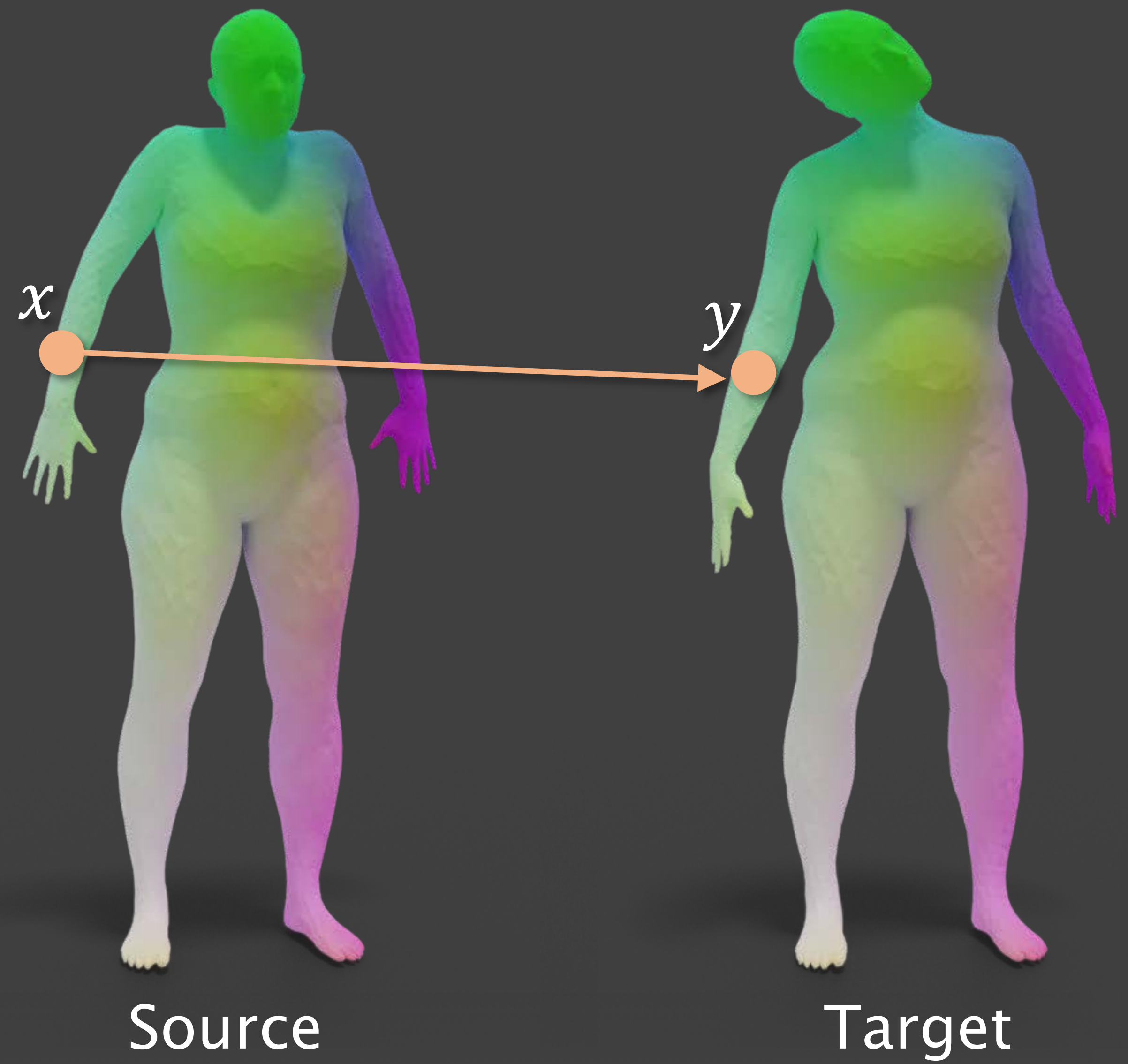
Structured Regularization of Functional Map Computations

Jing Ren, Mikhail Panine, Peter Wonka, Maks Ovsjanikov
KAUST, École Polytechnique

Shape Matching



- Point-based methods
  - [Bronstein et al. 2006],
  - [Huang et. Al 2008]…
- Parameterization-based methods
  - [Lipman and Funkhouser 2009]
  - [Aigerman et al. 2017]…
- Optimal transport
  - [Solomon et al. 2016]
  - [Mandad et al. 2017]…
- Functional maps
  - [Ovsjanikov et al. 2012]
  - [Ezuz and Ben-Chen 2017]…
- …

Functional map pipeline

Eigenfunctions of Laplace–Beltrami Operator

Helmholtz equation

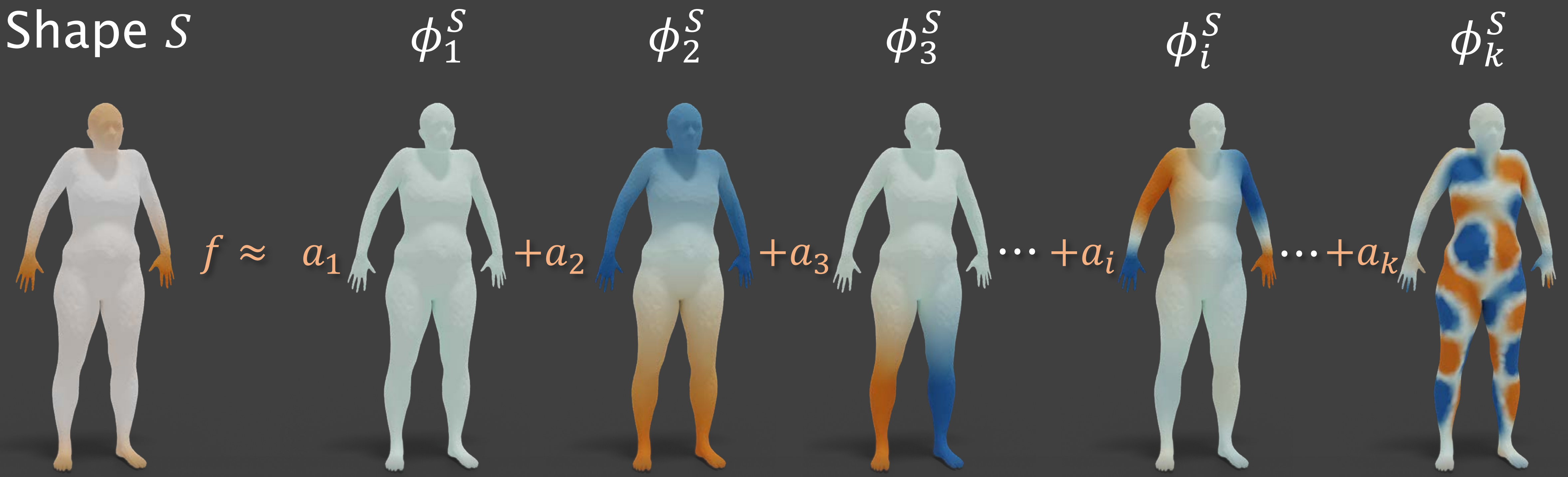
$$\Delta_S f = \lambda f$$

Shape S $\quad \phi_1^S \quad \phi_2^S \quad \phi_3^S \quad \phi_i^S \quad \phi_k^S$

$$0 = \lambda_1^S \leq \lambda_2^S \leq \lambda_3^S \leq \cdots \lambda_i^S \cdots \leq \lambda_k^S$$

Functional map pipeline

Function space basis

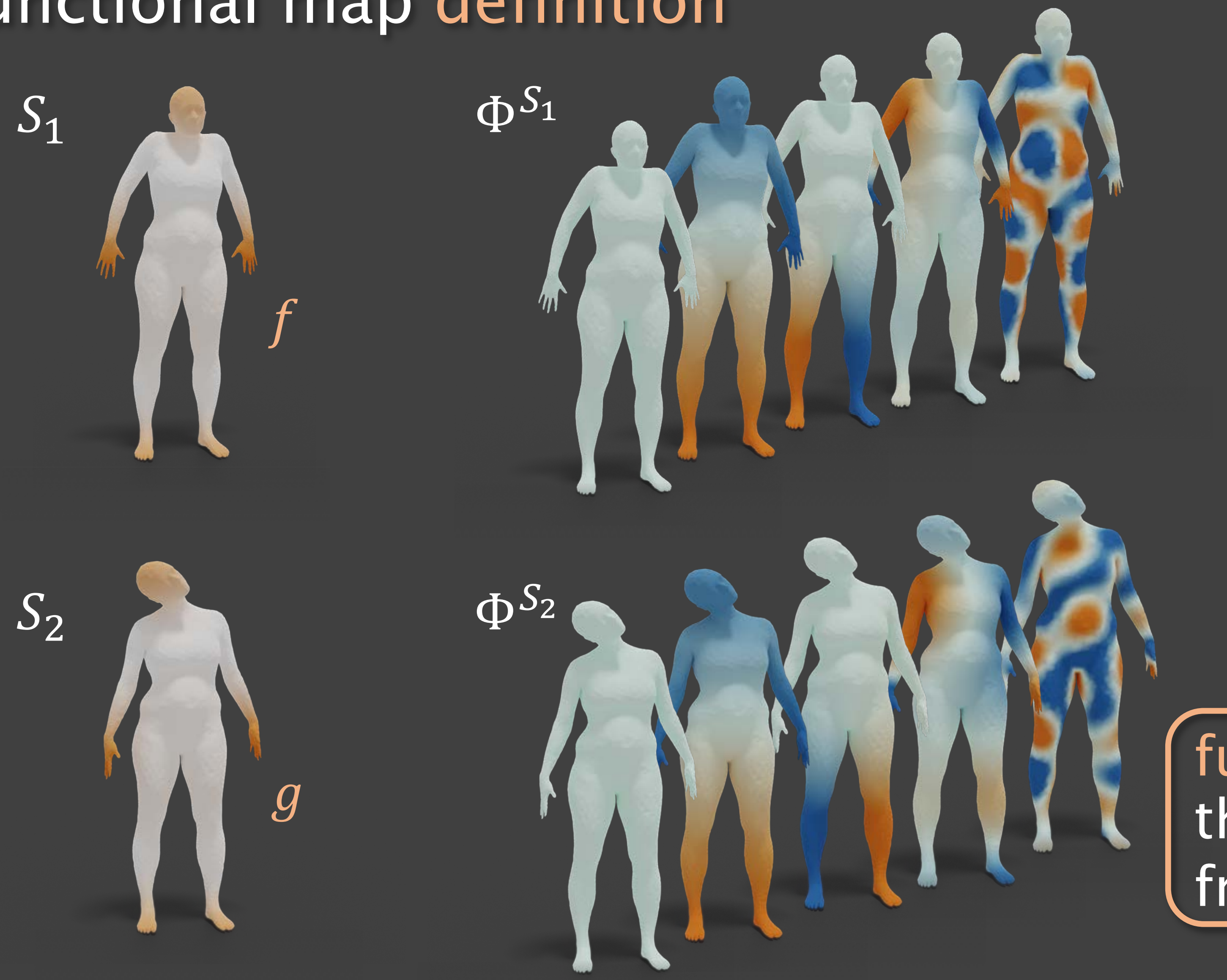


function f

$$f \approx a_1\phi_1^S + a_2\phi_2^S + \cdots a_k\phi_k^S = \Phi^S a$$

Functional map pipeline

Functional map definition



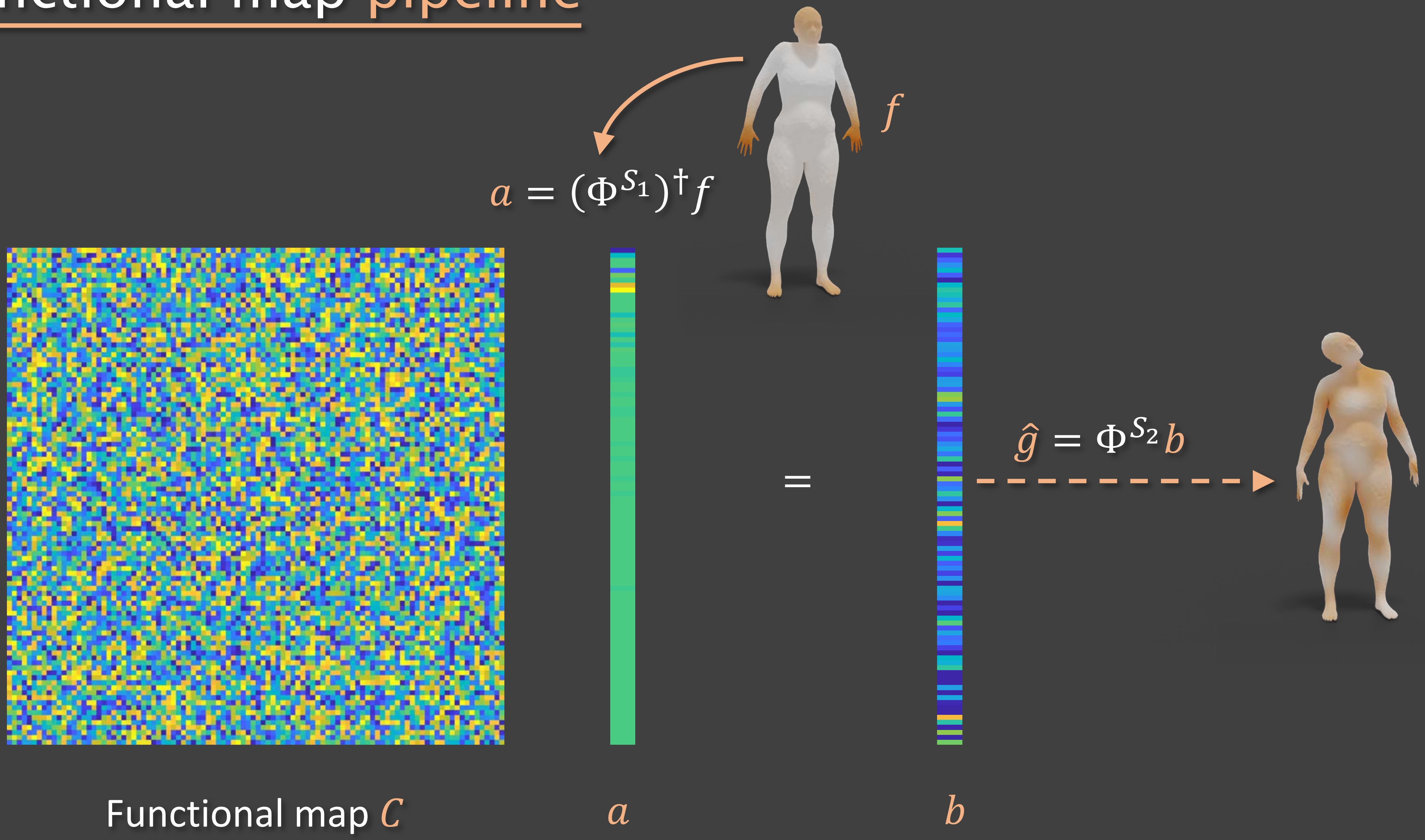
$$f \approx \Phi^{S_1} a$$

$$Ca = b$$

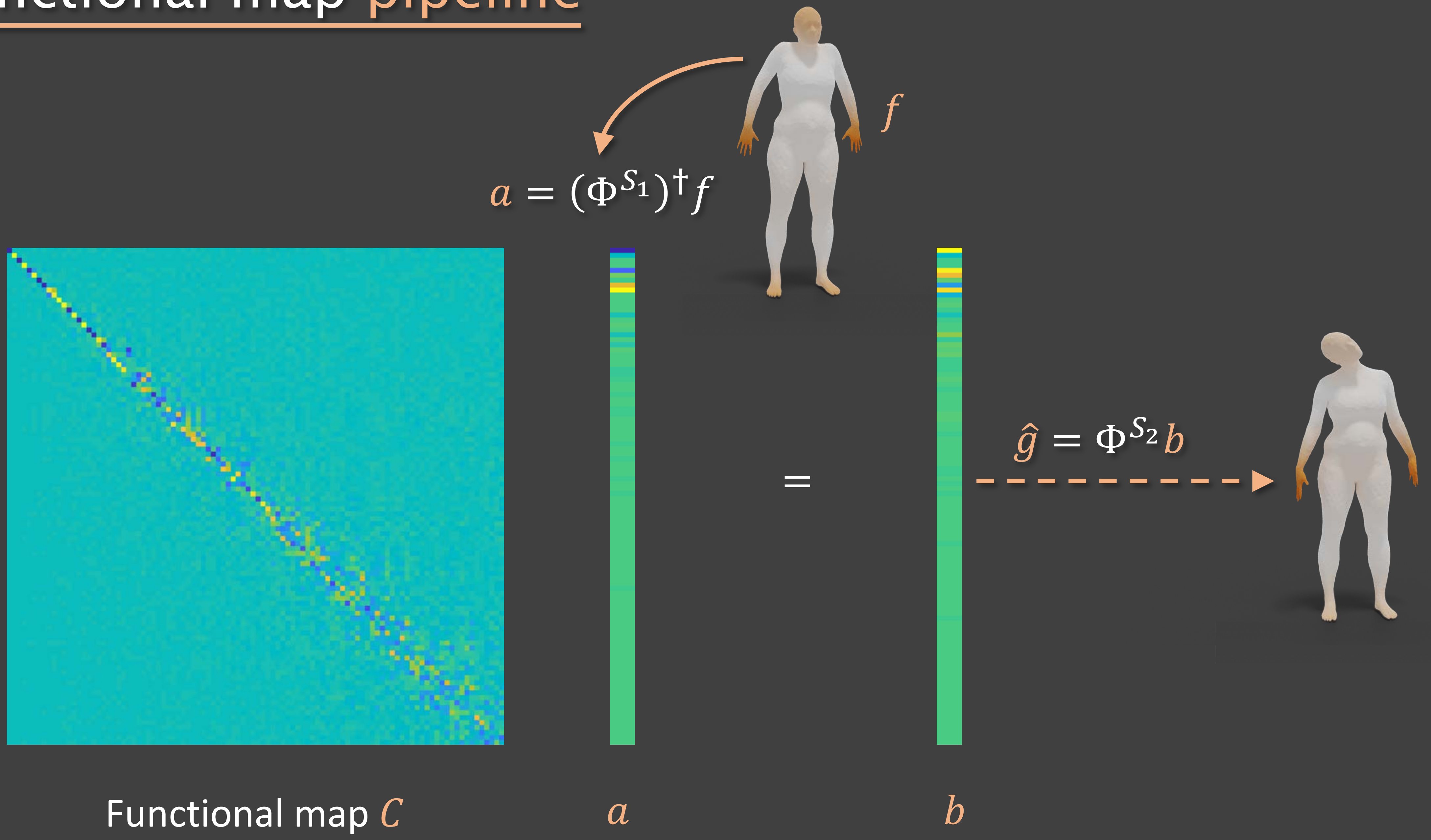
$$g \approx \Phi^{S_2} b$$

functional map: the matrix C that transports the coefficients from Φ^{S_1} to Φ^{S_2}

Functional map pipeline



Functional map pipeline



Functional map pipeline

$$C_{12}^{*} = \underset{C}{\operatorname{argmin}} \|CA - B\|_F^2$$

Descriptor preservation
[OBCS*12]

$$+w_1 \|C\Delta_1 - \Delta_2 C\|_F^2$$

Laplacian commutativity
[OBCS*12]

$$+w_2 \left\|C\Omega_1^{\text{multi}} - \Omega_2^{\text{multi}} C\right\|_F^2$$

Multiplicative operators
[NO17]

$$+w_3 \left\|C\Omega_1^{\text{orient}} - \Omega_2^{\text{orient}} C\right\|_F^2$$

Orientation preservation
[RPWO18]

$$+ \cdots$$

Outline

- Laplacian commutativity – widely used
- Drawbacks of the standard Laplacian commutativity
  - Unbounded in the smooth setting
  - Not aligned with the ground-truth functional map
- Propose the resolvent Laplacian commutativity
  - Bounded operator
  - Better aligned
- Quantitative results
  - Better stability
  - Better accuracy

Reformulate the Laplacian-Commutativity term

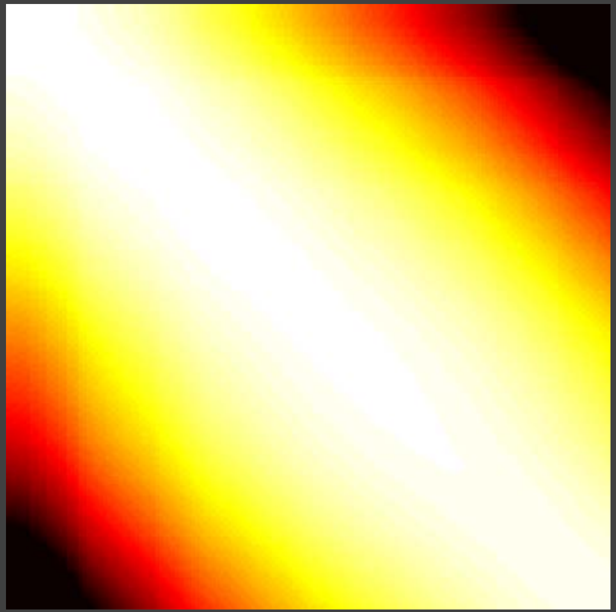
$$E(C) = \|C\Delta_1 - \Delta_2 C\|_F^2$$

$$= \|C\mathrm{diag}(\Lambda_1) - \mathrm{diag}(\Lambda_2)C\|_F^2$$

$$= \sum_{(i,j)} M_{ij} C_{ij}^2$$

where $M_{ij} = \left(\lambda_j^{S_1} - \lambda_i^{S_2}\right)^2$

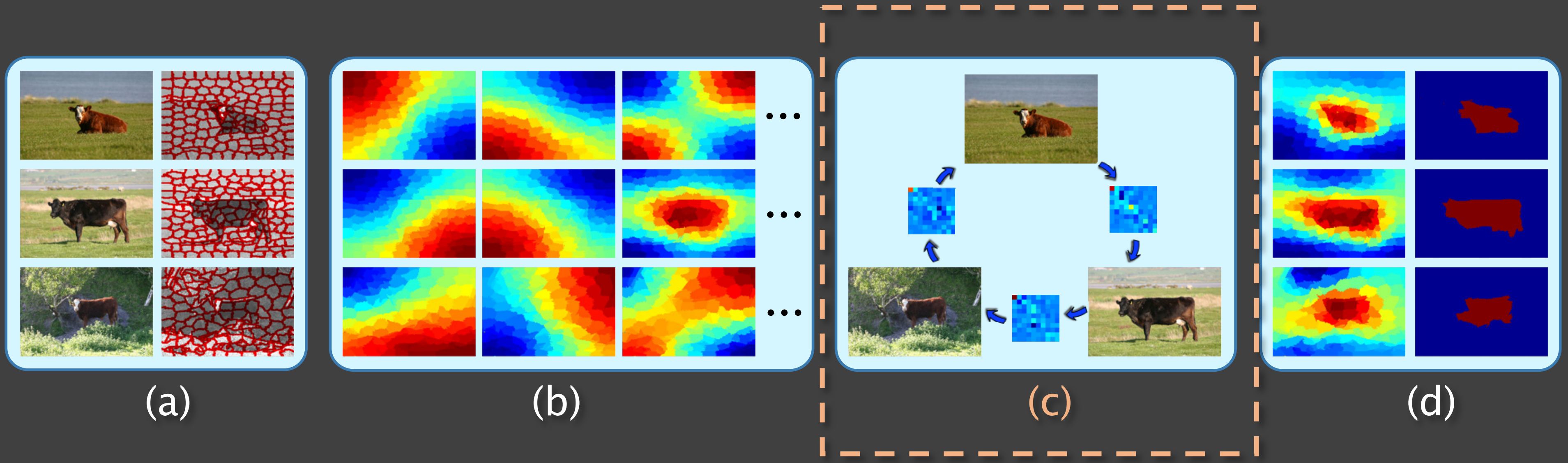
Mask M

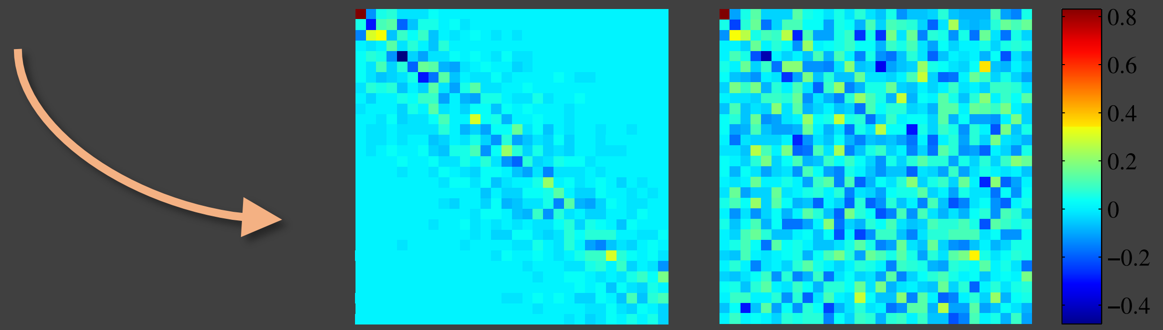


Applications of the Laplacian commutativity

"Image Co-Segmentation via Consistent Functional Maps"

Fan Wang, Qixing Huang, Leonidas J. Guibas

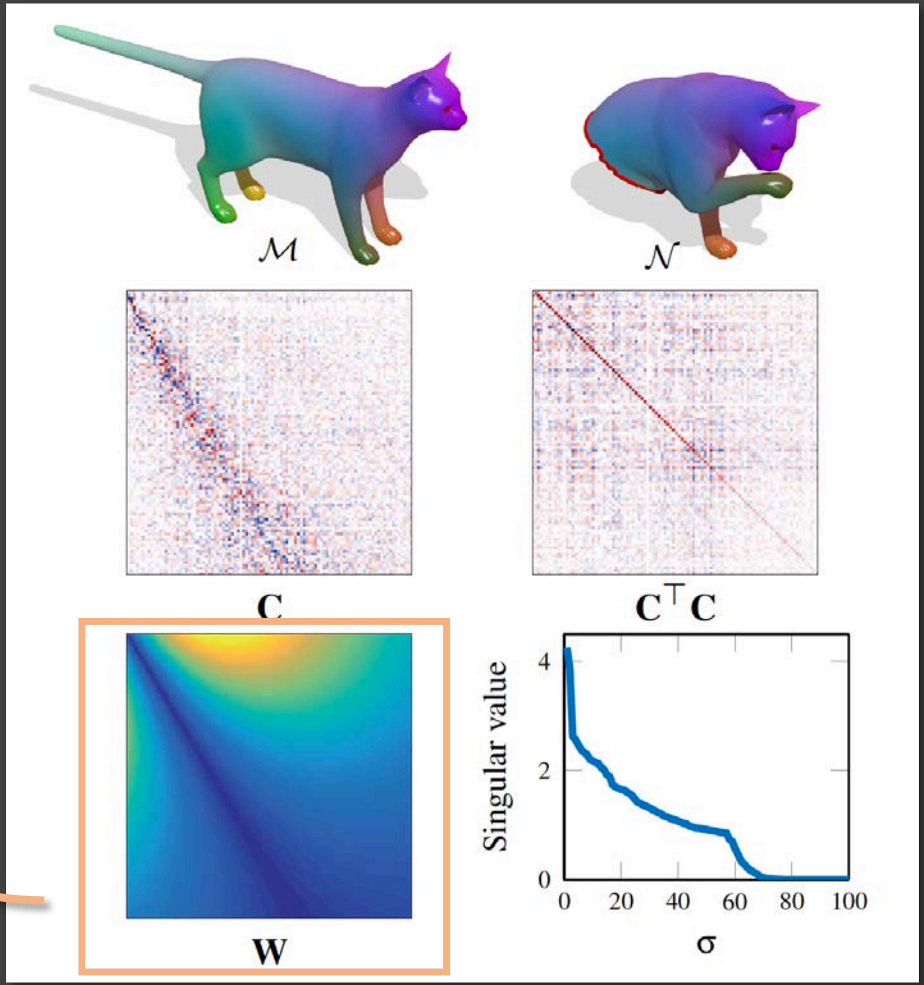




Applications of the Laplacian commutativity

"Partial Functional Correspondence"
E. Rodolà , L. Cosmo, M.M. Bronstein, A.Torsello, D. Cremers

$$\rho_{\mathrm{corr}}(C) = \sum_{ij} W_{ij} C_{ij}^2 + \cdots$$



Drawbacks of the Laplacian commutativity

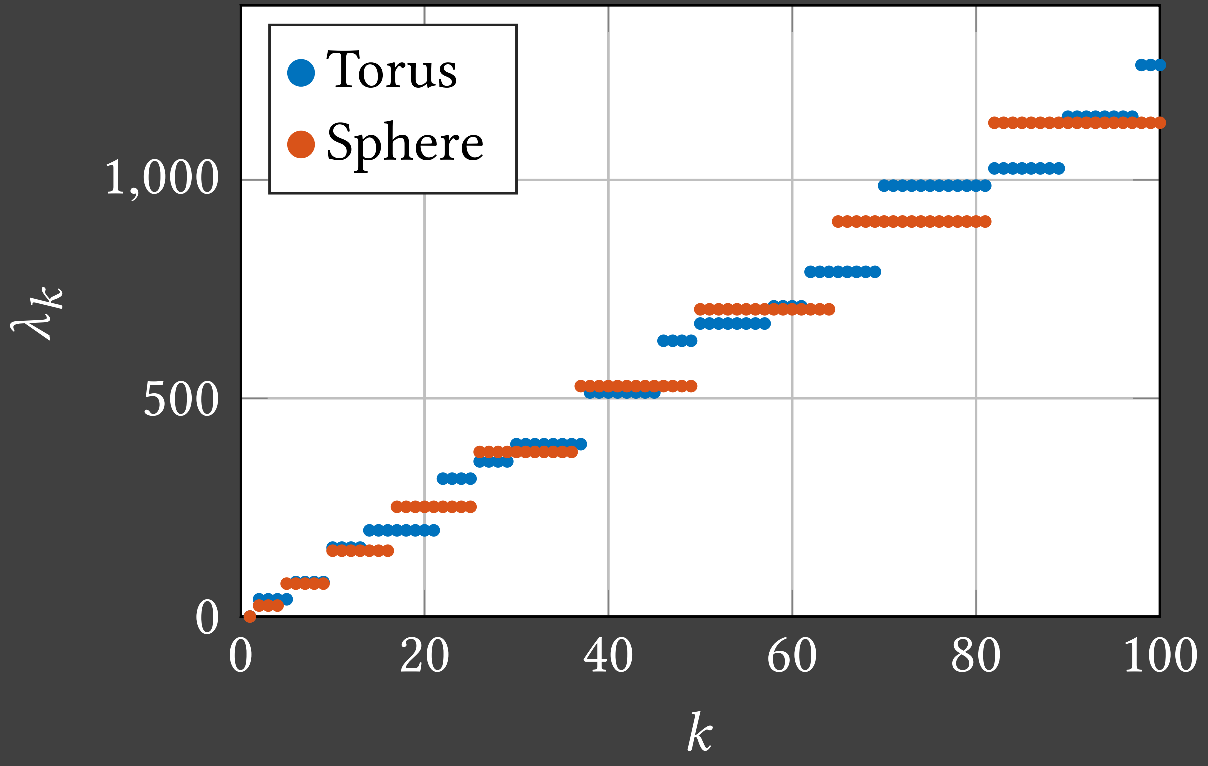
- Unboundedness
  - in the full LB basis (of smooth manifolds)

$$\|C_{12}\Delta_1 - \Delta_2 C_{12}\|^2 \to \infty$$

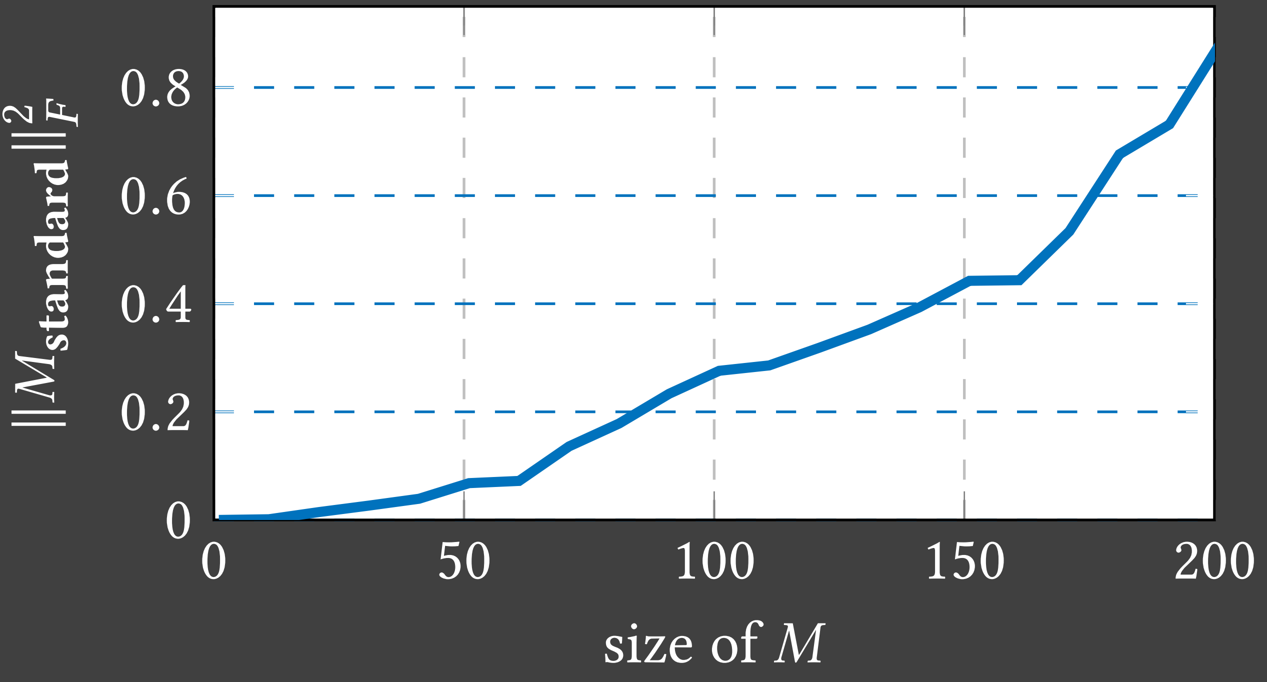
- Structure misalignment

Unboundedness Example

Spectrum of torus and sphere with unit area



$\|M_{\text{standard}}\|_F^2$ v.s. increasing size of M_{standard}



Unboundedness Example

$S_2 : \Delta_2 = c\Delta_1$

$c \neq 1$

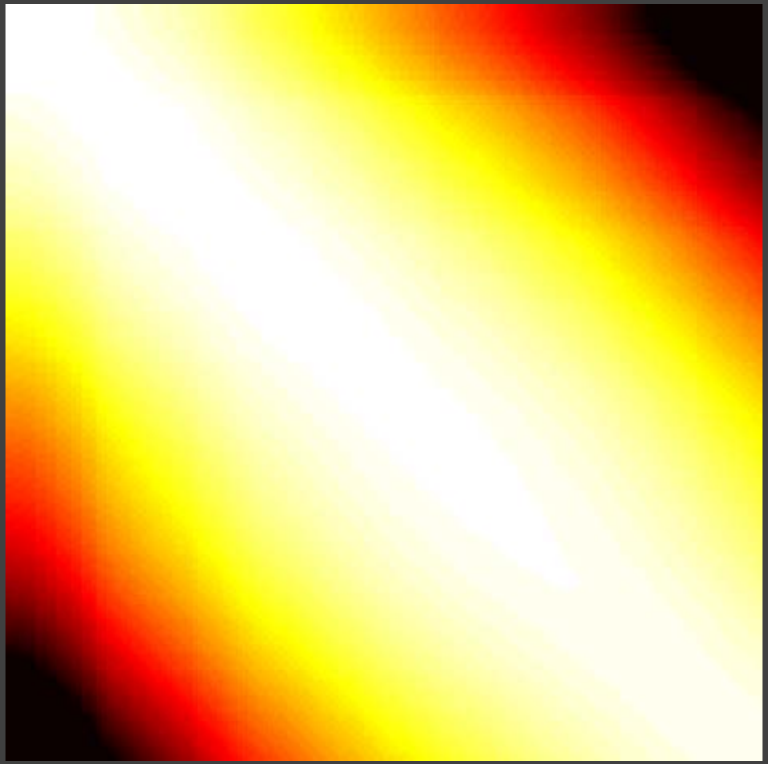
$S_1 : \Delta_1$

$$||C_{12}\Delta_1 - \Delta_2 C_{12}||^2 = (c-1)^2 ||\Delta_1||_F^2$$

$$\rightarrow \infty$$

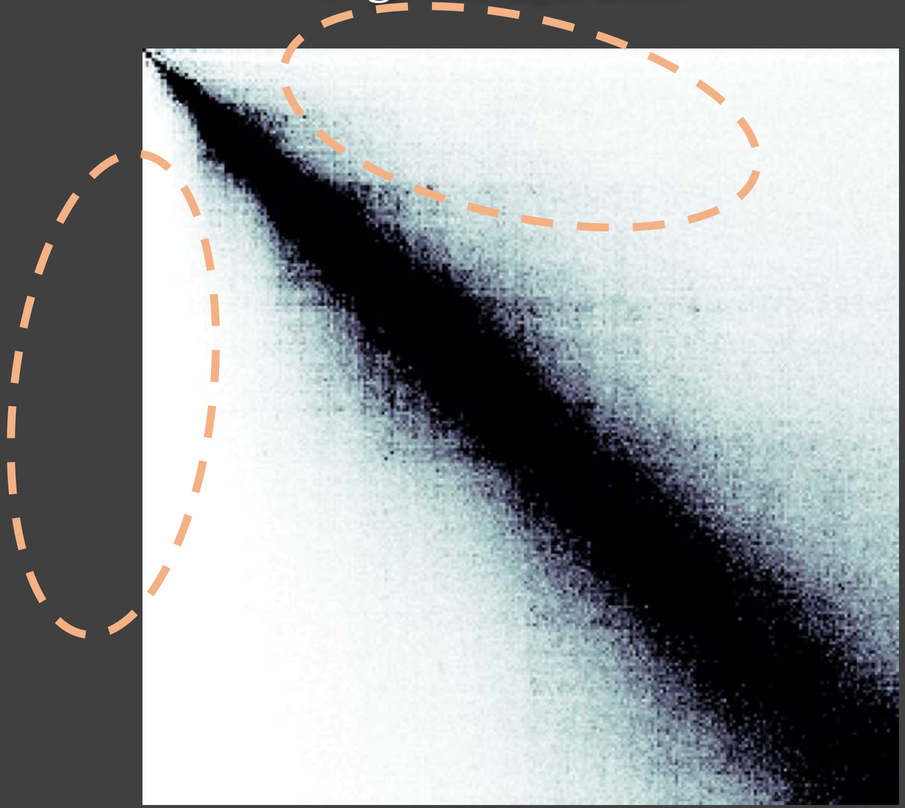
Structure misalignment

Mask M_{standard}



where $M_{ij} = \left(\lambda_j^{S_1} - \lambda_i^{S_2}\right)^2$

$(C_{\text{ground\_truth}})^2$



Funnel-shape

Our solution

- Boundedness: $\Delta \rightarrow$ resolvent of Δ
- Structure alignment: $\Delta \rightarrow \Delta^{\gamma}$

Resolvent operator

Definition

Let A be a possibly unbounded linear operator (with some technical assumption), the resolvent of A at μ is defined as

$$R_\mu(A) = (A - \mu I)^{-1}$$

- μ is a complex number
- $R_\mu(A)$ is defined for all μ NOT in the spectrum of A

$R_{a+ib}(\Delta)$ is well-defined for any $(a + ib)$ NOT in the non-negative real line (which contains the spectrum of Δ)

Resolvent operator

Applications

Important tool in operator theory

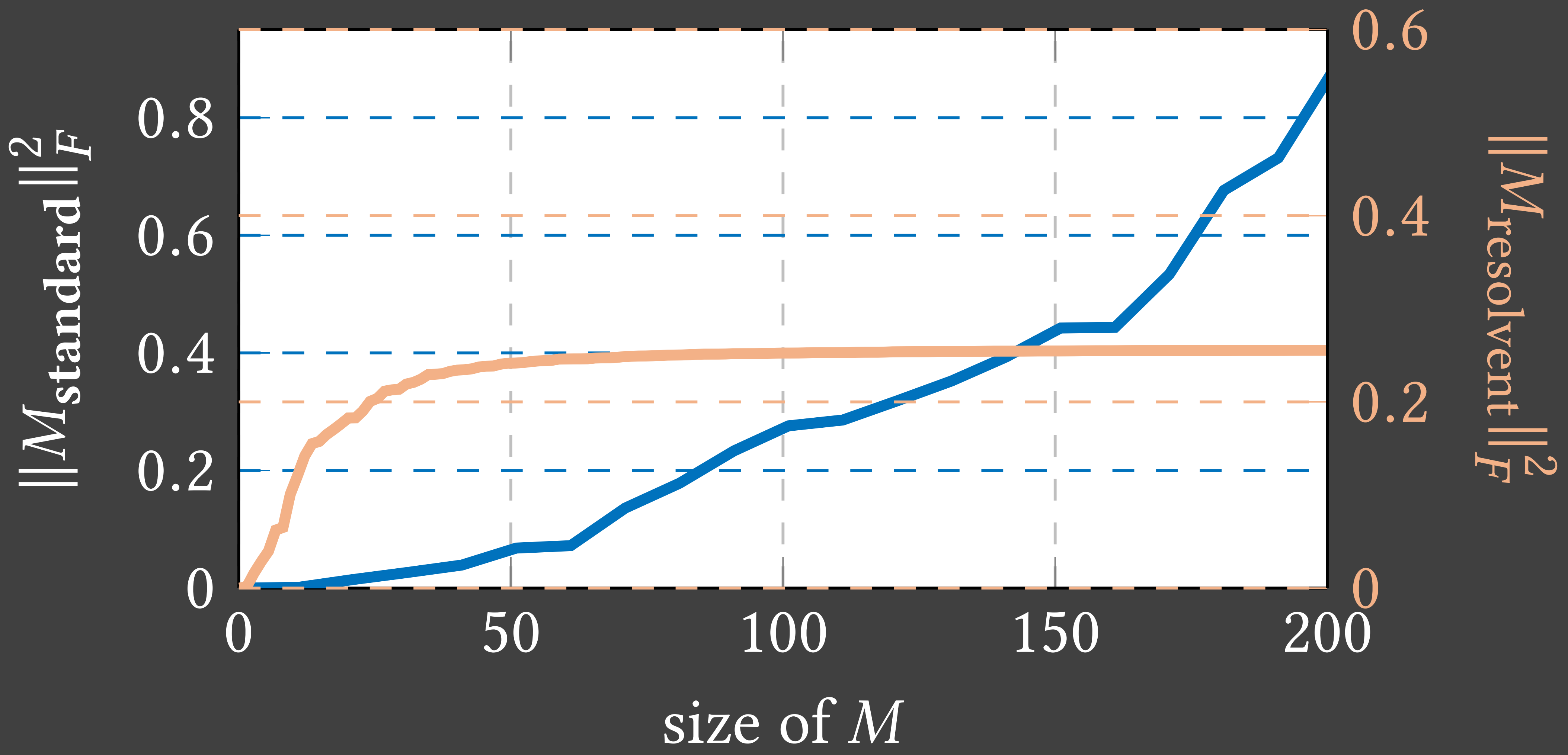
- Spectral theory: used in the definition of spectrum
- Unbounded self-adjoint operators: norm-resolvent convergence $d(A,B) = \|R_\mu(A) - R_\mu(B)\|$

Bounded resolvent Laplacian–Commutativity

Theorem 1 (Bounded Resolvent Commutativity) Let C_{12} be a bounded functional map. Then in the operator norm,

$$\left\| C_{12} R\left(\Delta_1^{\gamma}\right) - R\left(\Delta_2^{\gamma}\right) C_{12} \right\|_{\mathrm{op}}^2 < \infty$$

Bounded resolvent Laplacian-Commutativity



Bounded resolvent Laplacian-Commutativity

- $\Delta \rightarrow$ standard Laplacian commutator
- $R_{a+ib}(\Delta^{\gamma})$: well-defined and bounded
  - Introduce γ to tune the structure of the mask
  - Our resolvent Laplacian commutator

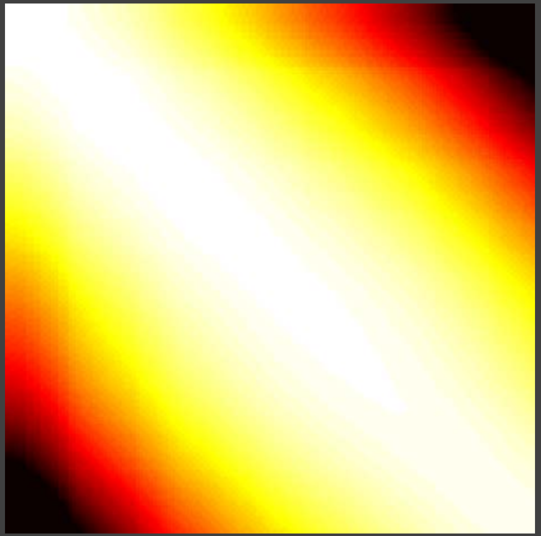
$$E(C_{12}) = \sout{\|C_{12}\Delta_1 - \Delta_2 C_{12}\|_F^2} = \left\|C_{12}R\left(\Delta_1^{\gamma}\right) - R\left(\Delta_2^{\gamma}\right)C_{12}\right\|_F^2$$

Resolvent mask

* Def: $R_\mu(A) = (A - \mu I)^{-1}$

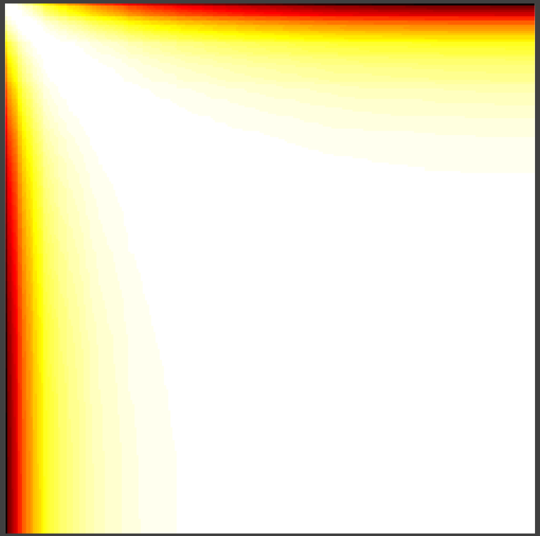
- Δ has eigenvalues λ_k
- $R_i(\Delta^{1/2})$ has eigenvalues $r_k = \dfrac{1}{\sqrt{\lambda_k} - i} = \dfrac{\sqrt{\lambda_k}}{\lambda_k + 1} + \dfrac{1}{\lambda_k + 1} i$

Mask M_{standard}



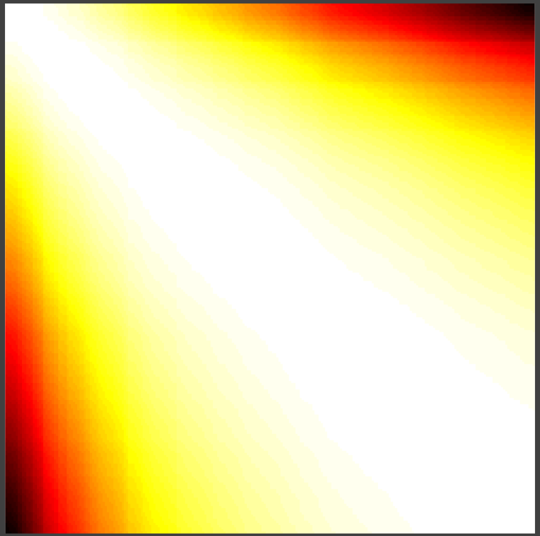
$$M_{ij} = \left(\lambda_j^{S_1} - \lambda_i^{S_2}\right)^2$$

Real part



$$M_{ij}^{\text{Re}} = \left(\frac{\sqrt{\lambda_j^{S_1}}}{\lambda_j^{S_1}+1} - \frac{\sqrt{\lambda_i^{S_2}}}{\lambda_i^{S_2}+1}\right)^2$$

Imaginary part

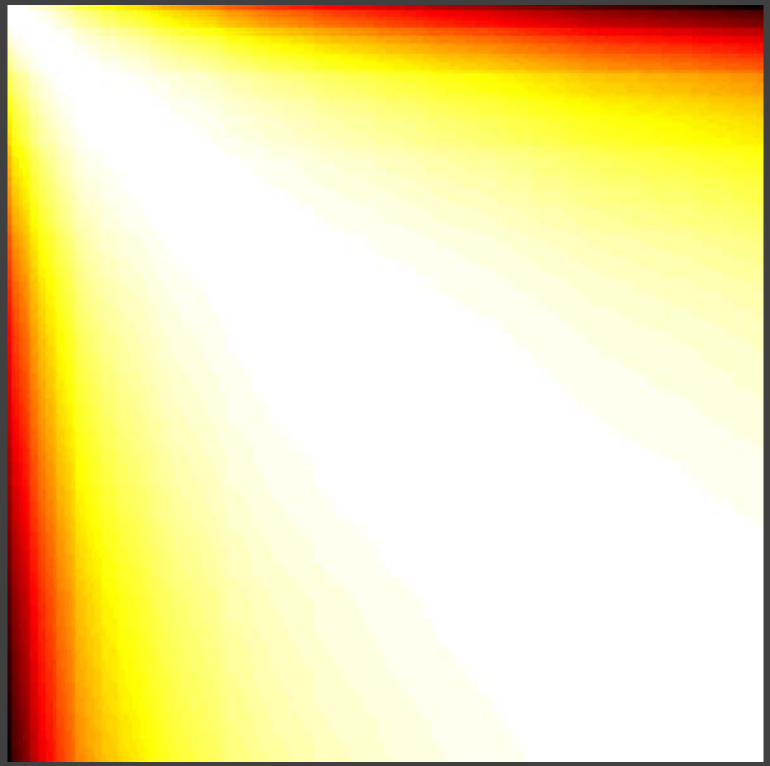


$$M_{ij}^{\text{Im}} = \left(\frac{1}{\lambda_j^{S_1}+1} - \frac{1}{\lambda_i^{S_2}+1}\right)^2$$

Resolvent mask

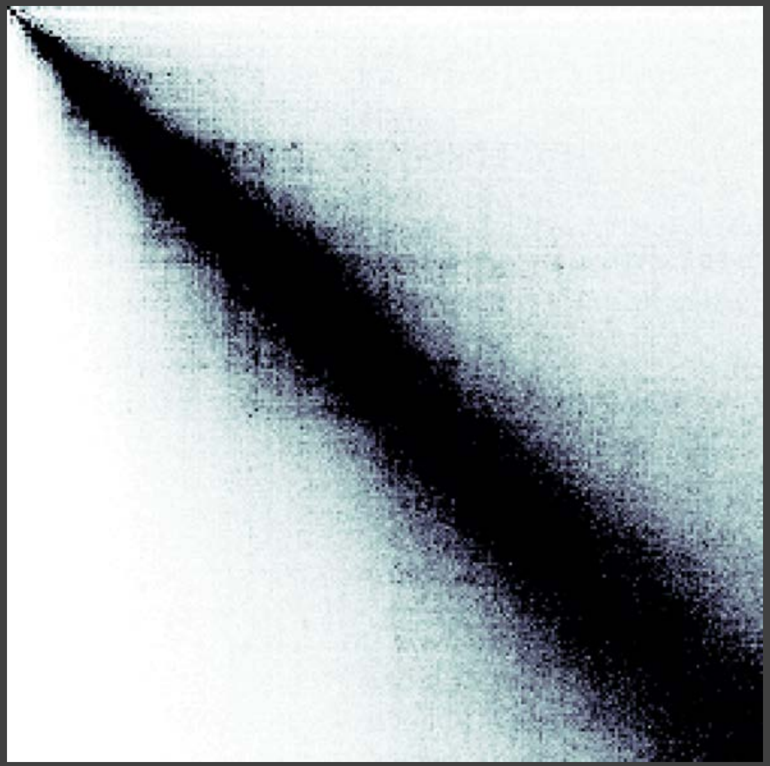
$$\left\|C_{12}R(\Delta_1^\gamma) - R(\Delta_2^\gamma)C_{12}\right\|_F^2 = \sum_{i,j} M_{ij}C_{12}^2$$

Mask $M_{\text{resolvent}}$



where $M_{ij} = M_{ij}^{\text{Re}} + M_{ij}^{\text{Im}}$

$(C_{\text{ground\_truth}})^2$

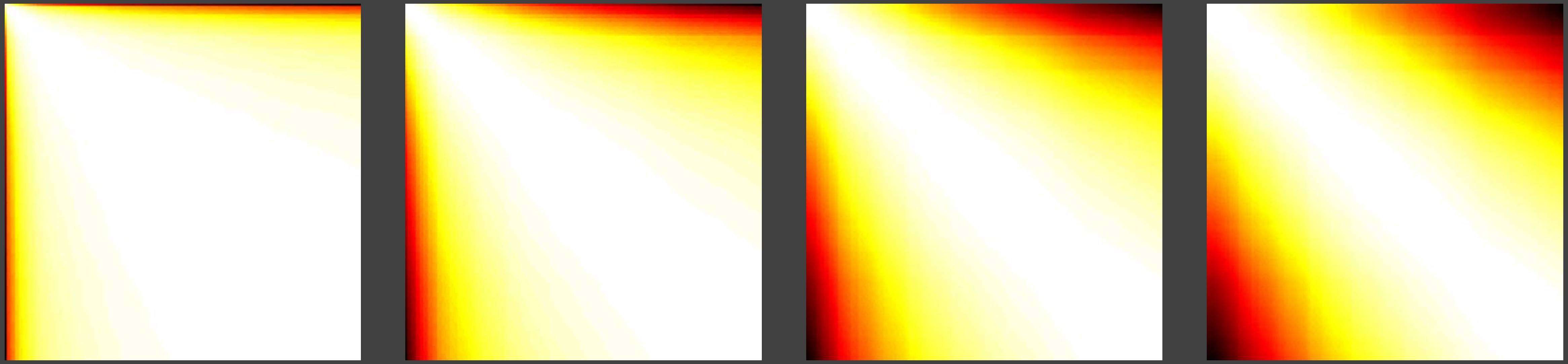


Funnel-shape

Mask reformulation of the resolvent commutativity

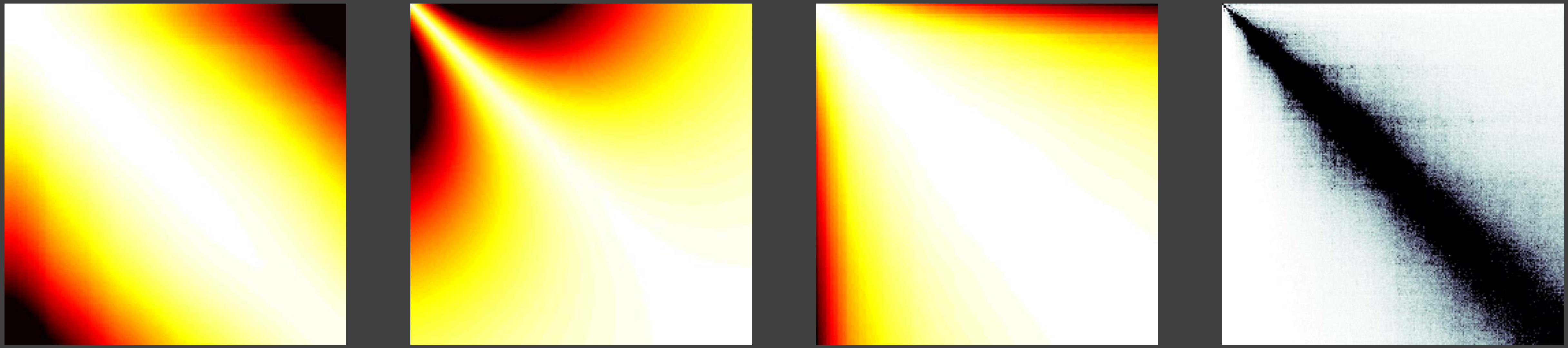
$$E(C_{12}) = \left\| C_{12} R(\Delta_1^{\gamma}) - R(\Delta_2^{\gamma}) C_{12} \right\|_F^2 = \sum_{(i,j)} M_{ij} C_{12}^2$$

$\gamma = 0.25$ $\gamma = 0.5$ $\gamma = 0.75$ $\gamma = 1$



Penalty mask v.s. ground-truth functional map

Standard mask	Slanted mask	Resolvent mask $\gamma = 0.5$	Mean squared ground-truth

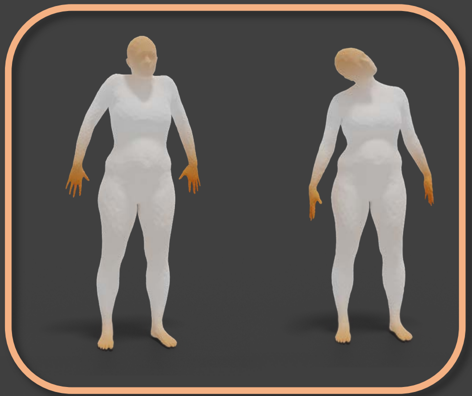


"Partial Functional Correspondences"
Rodolà et al

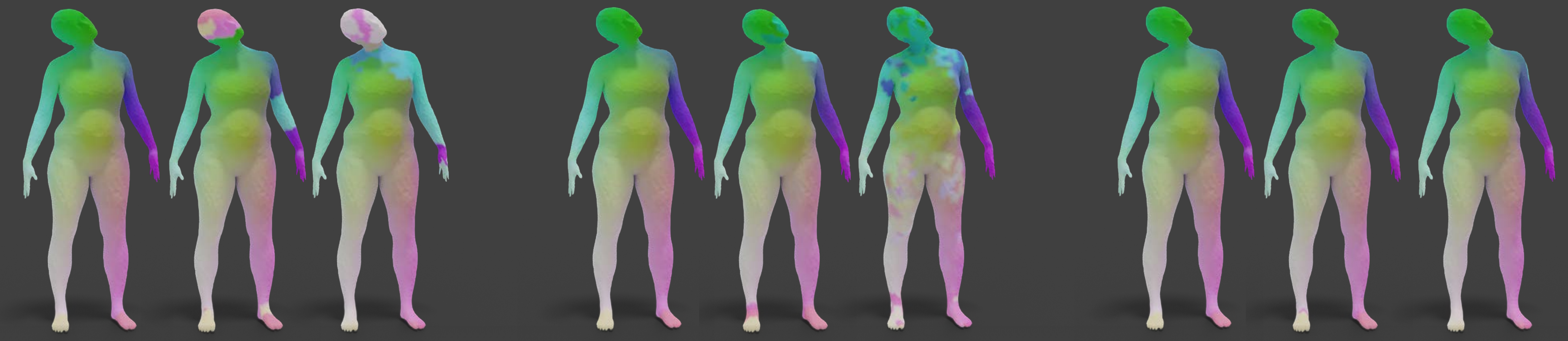
Results: Stability (example)

Source

Given one pair of descriptors
Compute a $k \times k$ functional map
k^2 variables!

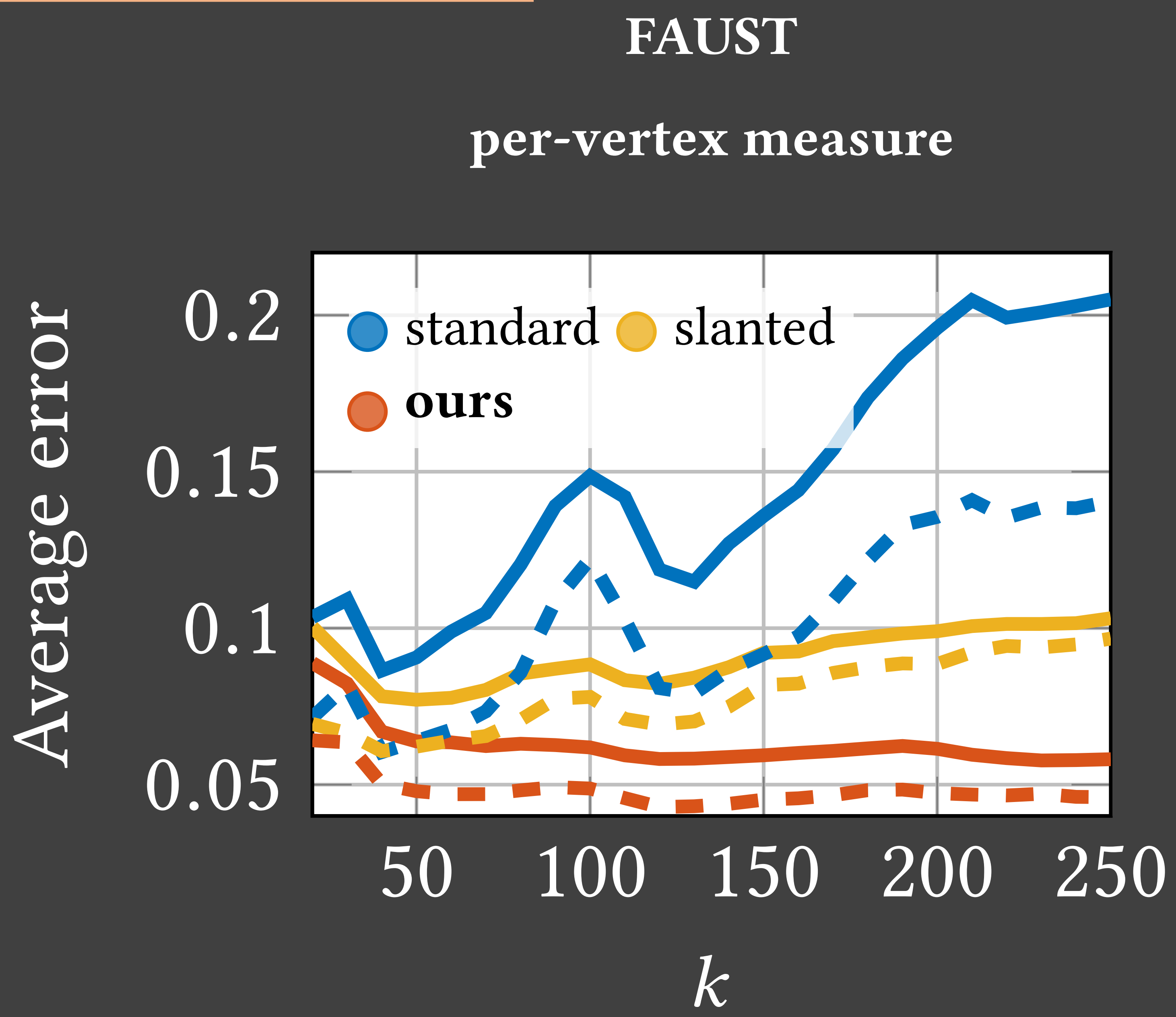


$k = 50$ $k = 100$ $k = 300$ $k = 50$ $k = 100$ $k = 300$ $k = 50$ $k = 100$ $k = 300$



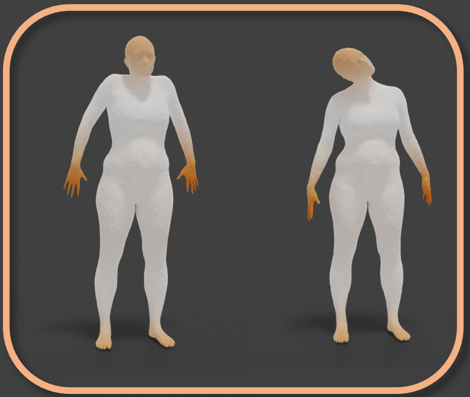
Standard Slanted Resolvent

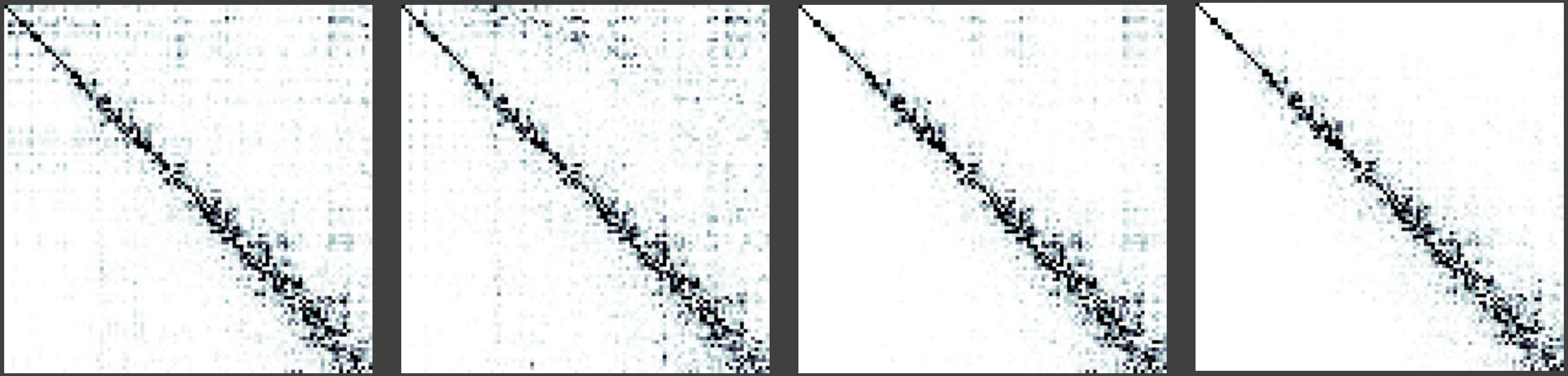
Results: Stability (summary)



Results: Accuracy (example)

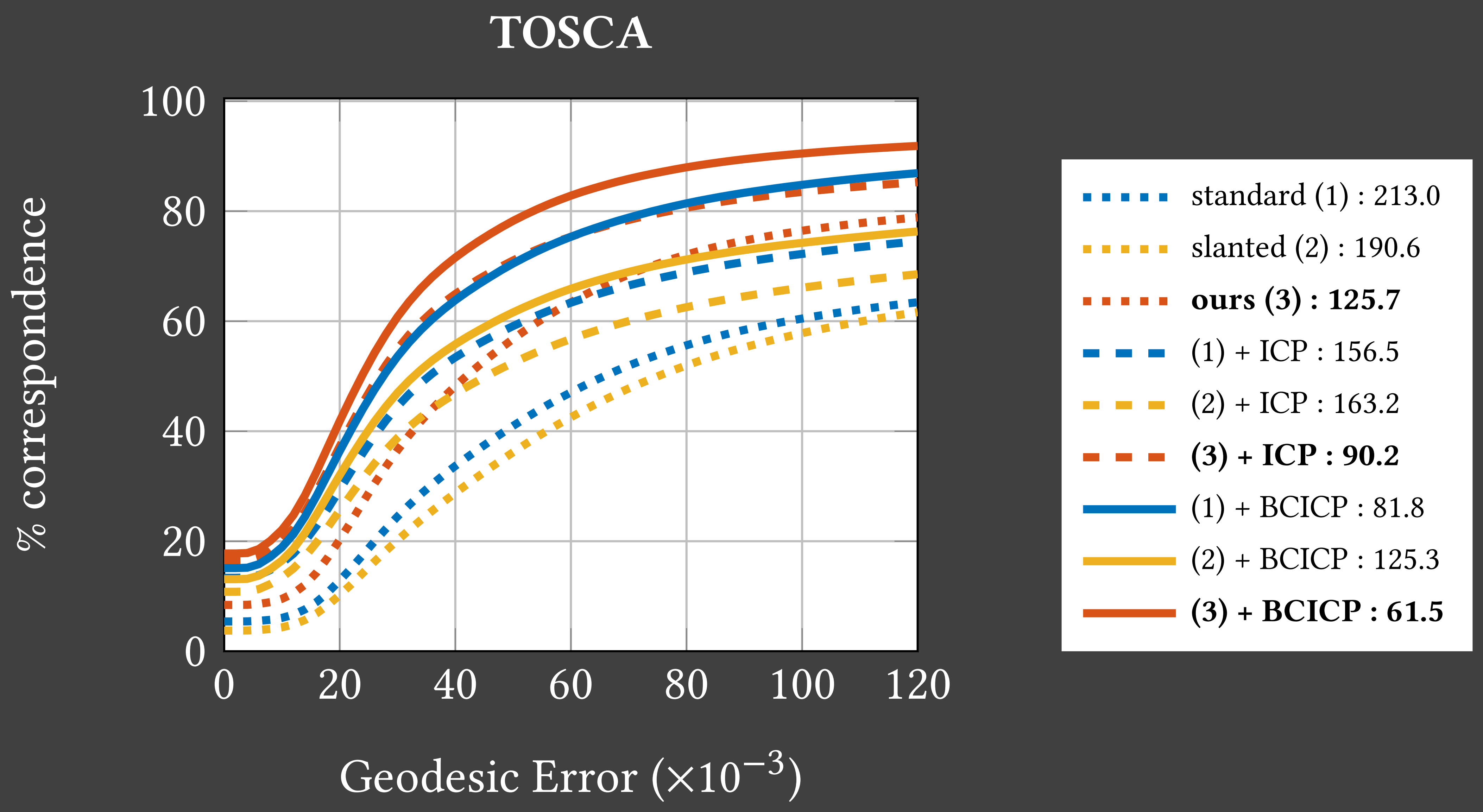
Given one pair of descriptors
Compute a 100×100 functional map



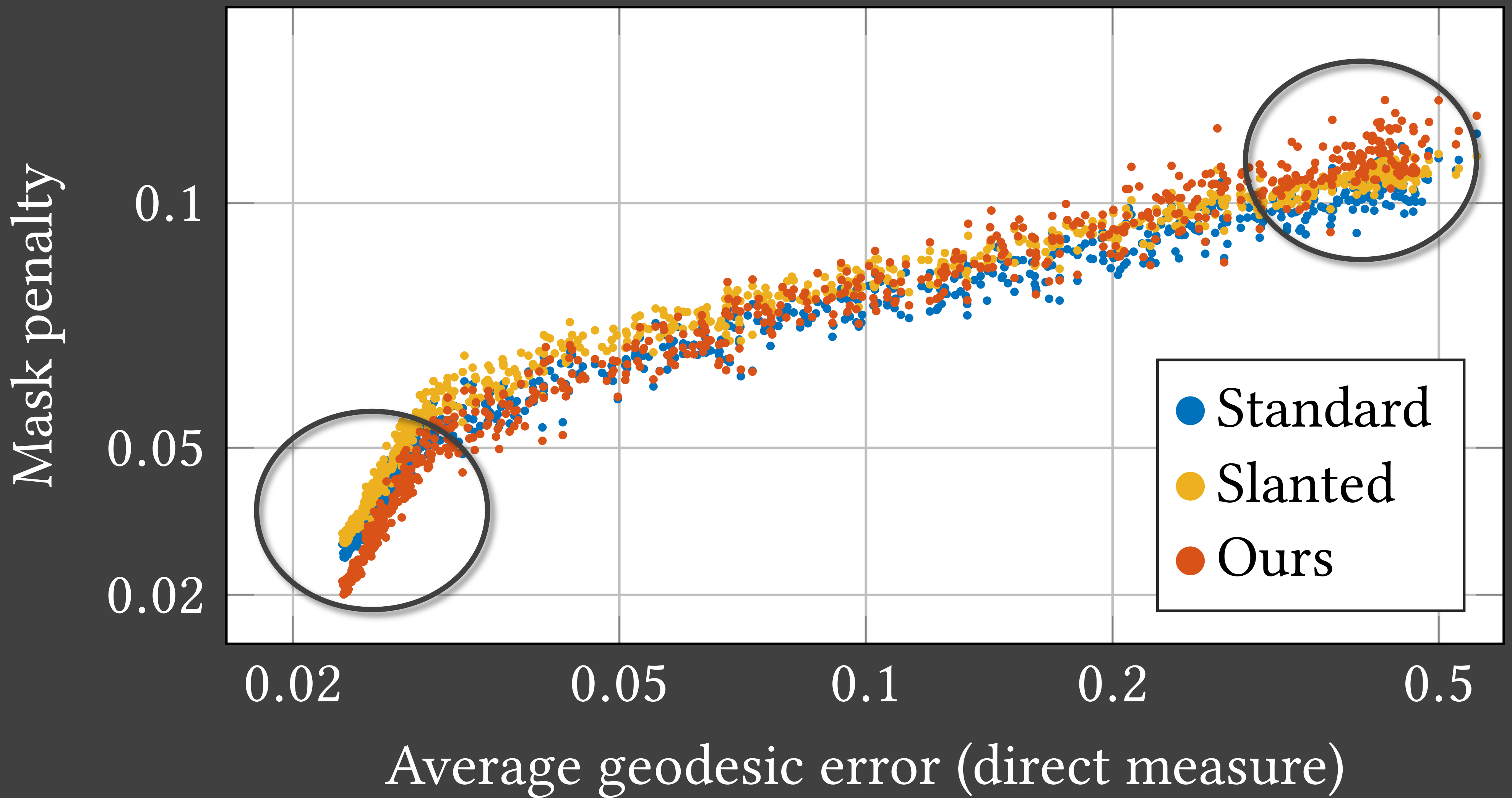


Standard | Slanted | Resolvent | Ground-truth

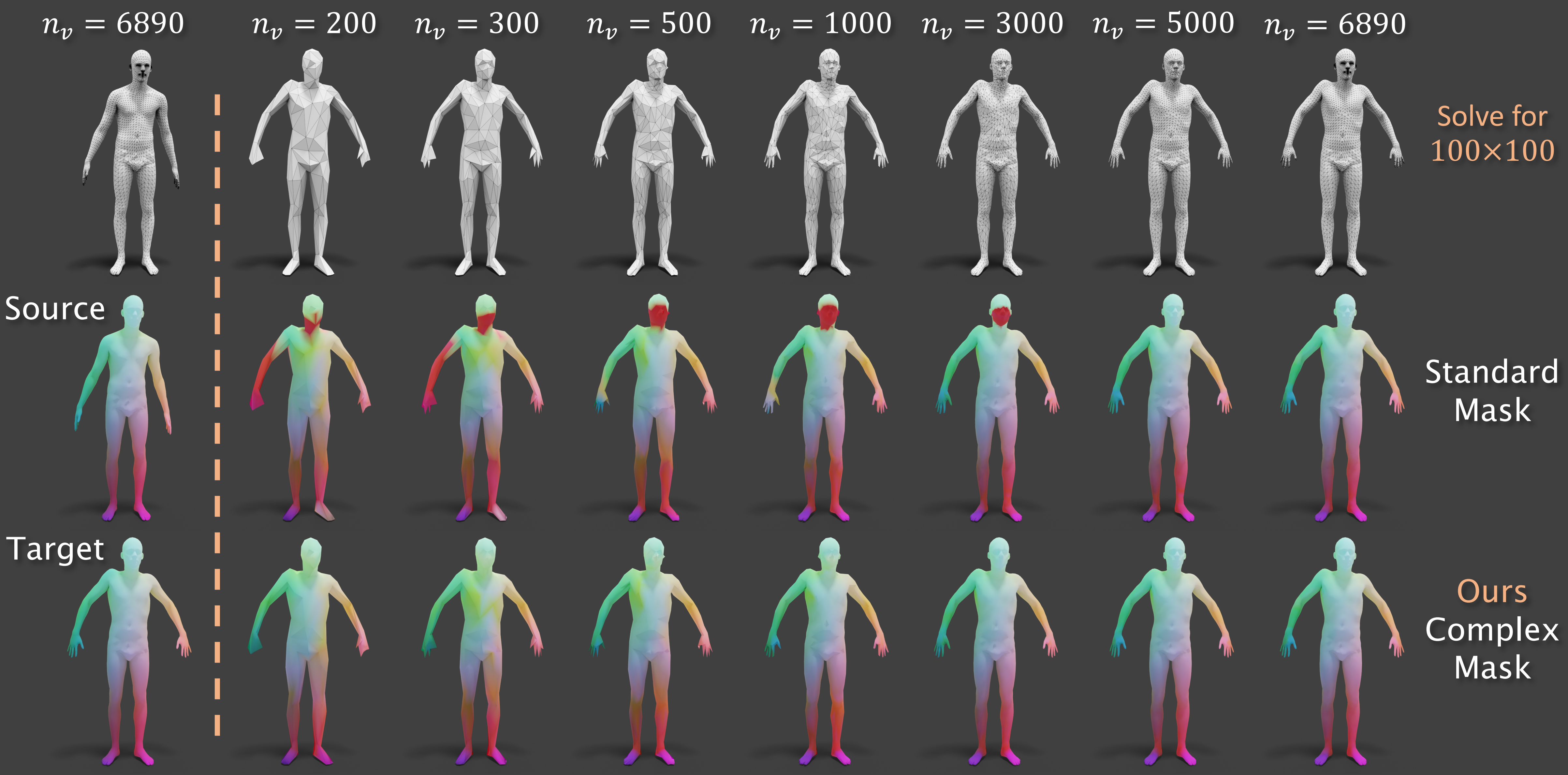
Results: Accuracy (summary)



Results: Correlation (fMap penalty v.s. pMap accuracy)



Results: Stability under remeshing and coarsening



Summary

- Shape matching – functional map pipeline
- Laplacian commutativity – widely used
- Drawbacks of the standard Laplacian commutativity
  - Unbounded in the smooth setting
  - Not aligned with the ground-truth functional map
- Propose the resolvent Laplacian commutativity
  - Bounded operator
  - Aligned with the funnel shape
- Results
  - Better accuracy
  - Better stability

Thanks for your attention

Structured Regularization of Functional Map Computations

Jing Ren, Mikhail Panine, Peter Wonka, Maks Ovsjanikov

KAUST, École Polytechnique

Sample code

Convergence the resolvent Laplacian

Lemma 2. Let Δ_1 and Δ_2 be Laplacians on compact, connected, oriented surfaces M_1 and M_2, respectively. Let $C_{12}: L_2(M_1) \to L_2(M_2)$ be a bounded operator. If $\gamma > \frac{1}{2}$, then:

$$\|C_{12} R_\mu(\Delta_1^\gamma) - R_\mu(\Delta_2^\gamma) C_{12}\|_{HS}^2 < \infty$$

Where μ is any complex number not on the non-negative real line.

Reformulate the Laplacian–Commutativity term

$$E(C_{12}) = \|C_{12}\Delta_1 - \Delta_2 C_{12}\|_F^2$$

$$= \|C_{12}\mathrm{diag}(\Lambda_1) - \mathrm{diag}(\Lambda_2)C_{12}\|_F^2$$

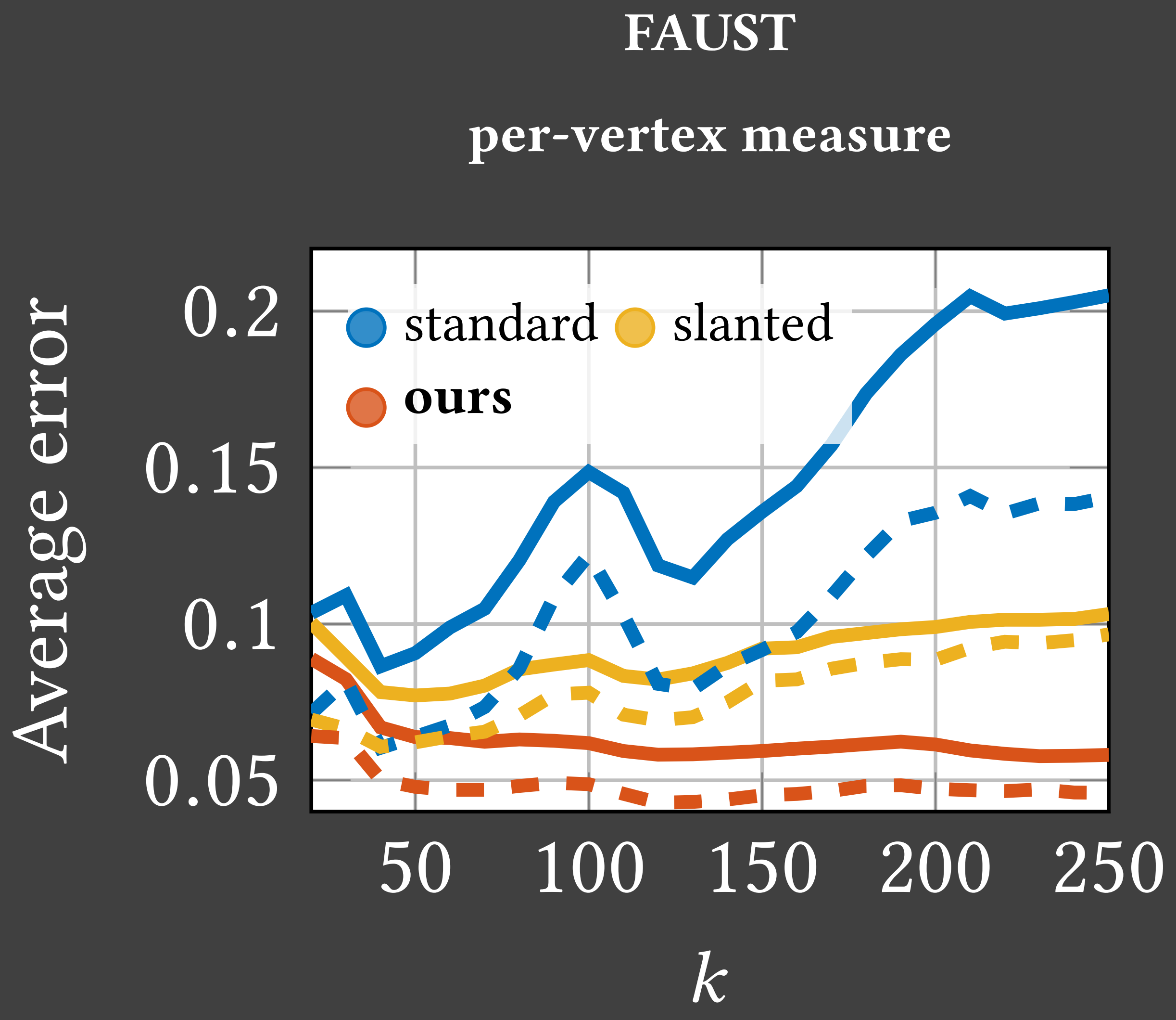
$$= \left\|C_{12}\otimes\left(1_{k_2}\Lambda_1^T\right) - \left(\Lambda_2 1_{k_1}^T\right)\otimes C_{12}\right\|_F^2$$

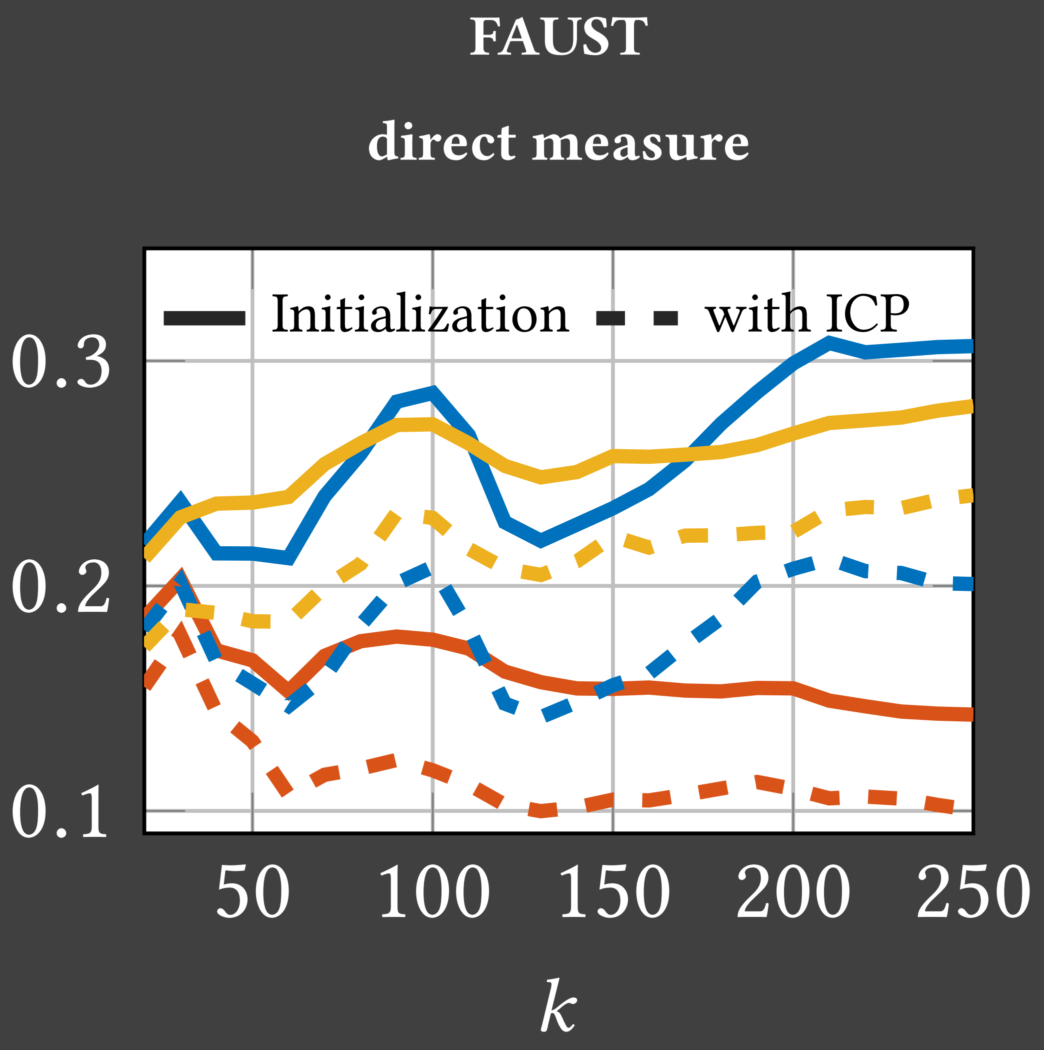
$$= \left\|\left(1_{k_2}\Lambda_1^T - \Lambda_2 1_{k_1}^T\right)\otimes C_{12}\right\|_F^2$$

$$= \sum_{(i,j)} M\otimes(C_{12})^2$$

Note: $\otimes$ is the entry-wise matrix multiplication

Results: Stability (summary)

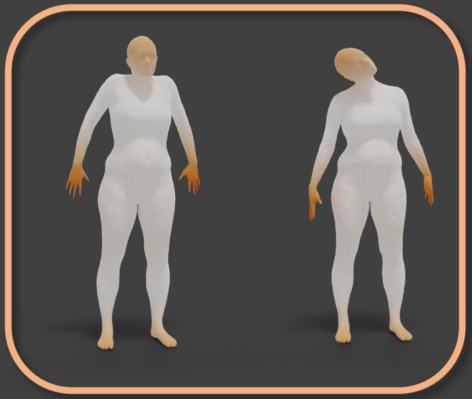


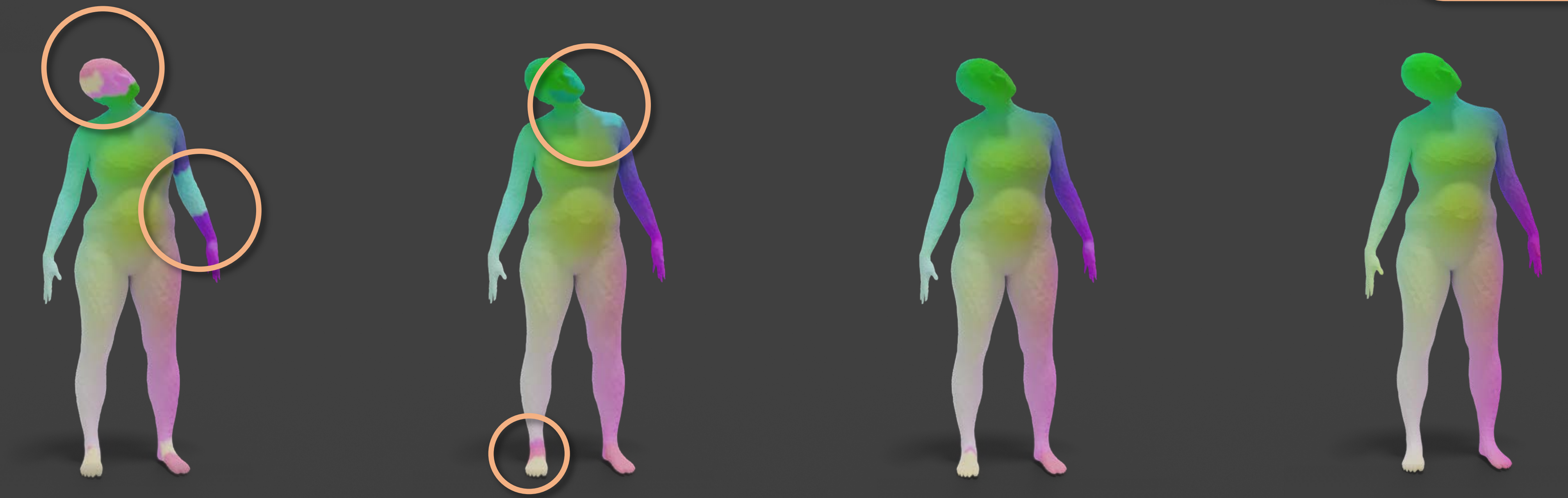


Results: Accuracy (example)

Source

Given one pair of descriptors
Compute a 100×100 functional map
Corresponding point-wise map

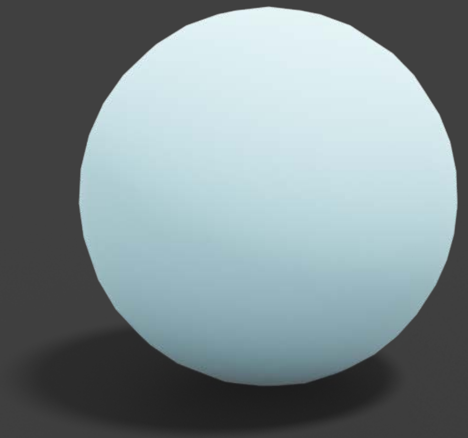




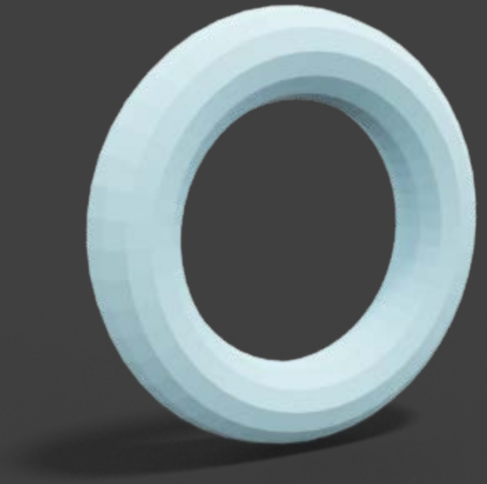
Standard   Slanted   Resolvent   Ground-truth

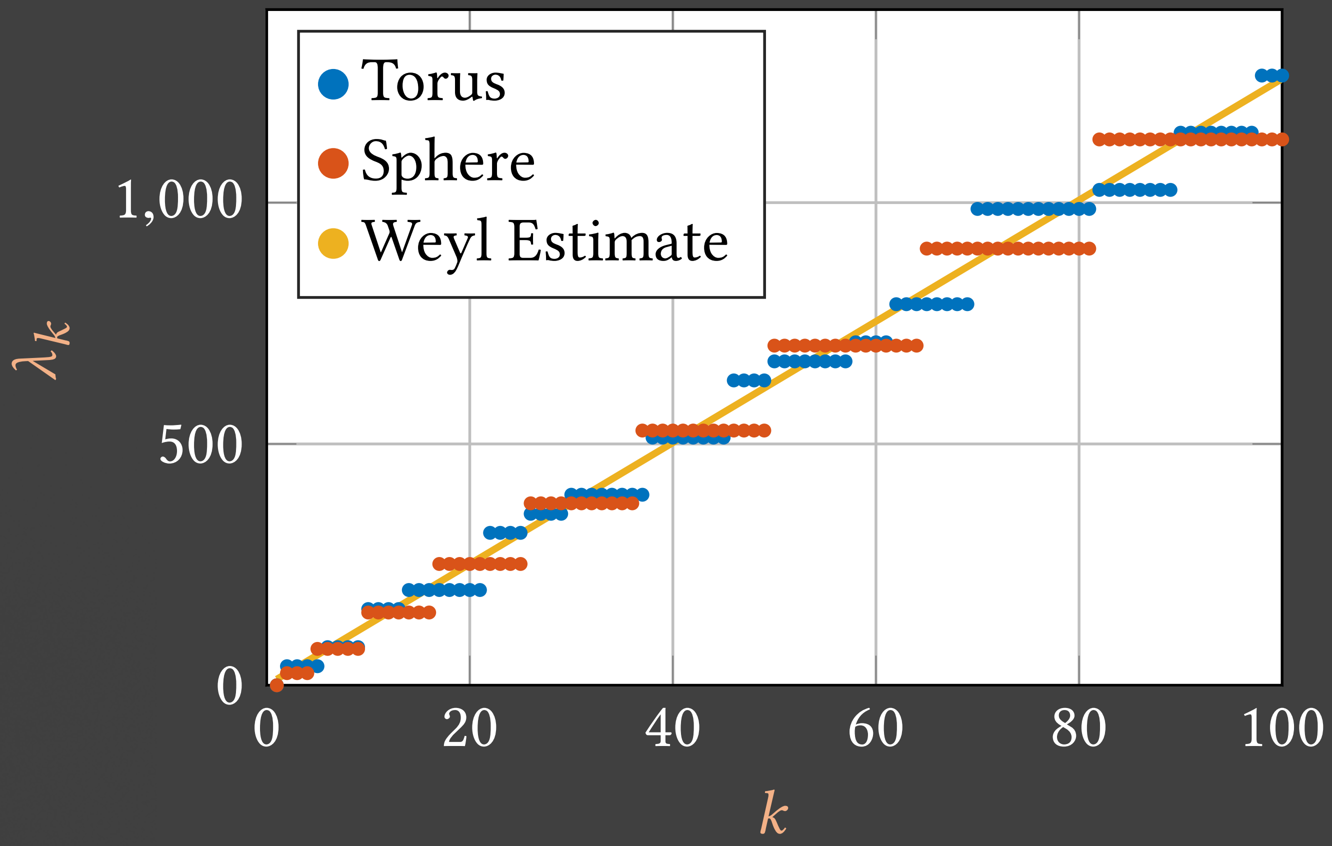
Unboundedness Example

Unit Sphere



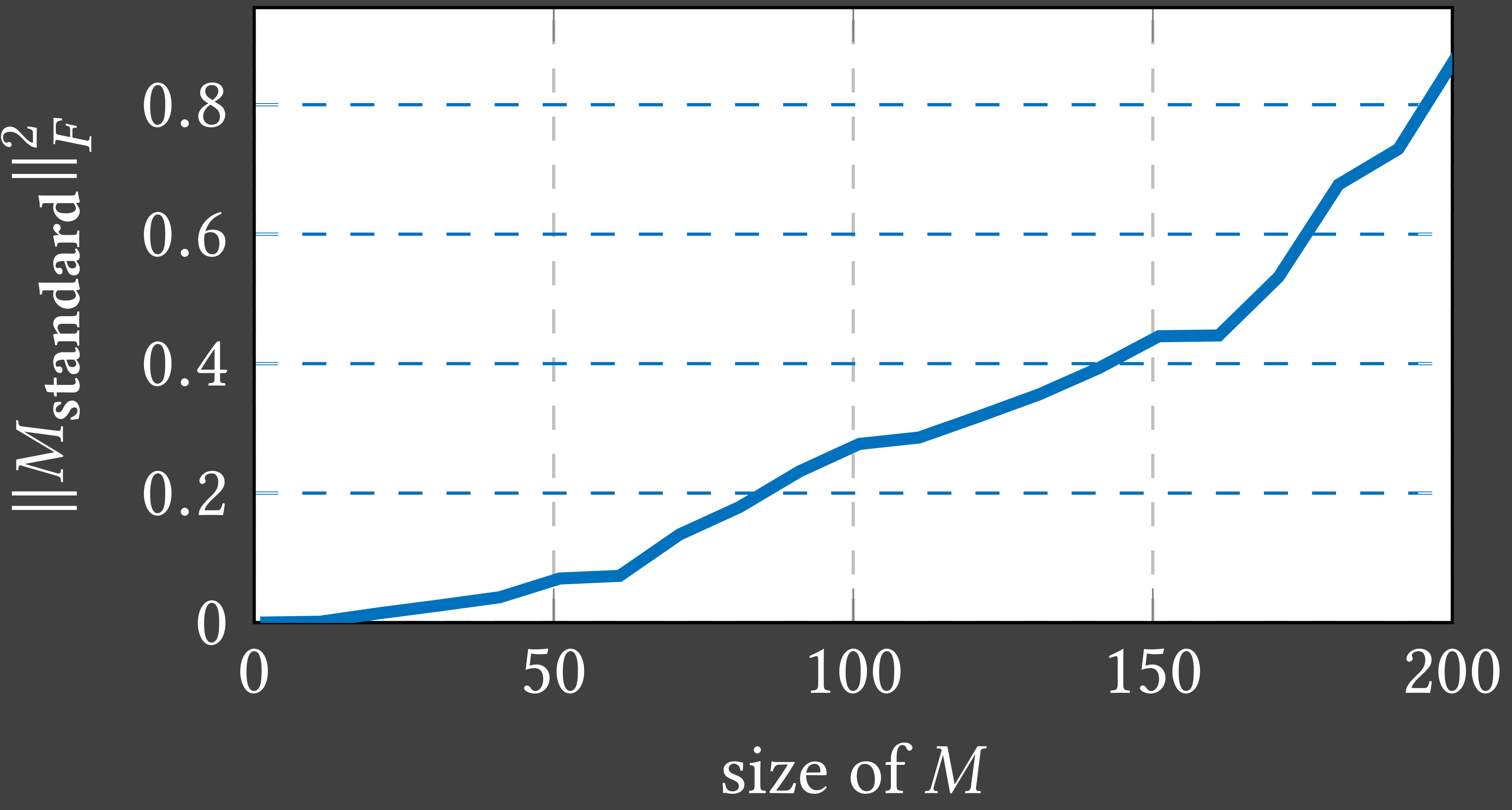
Unit Torus





Unbounded standard Laplacian-Commutativity

$\|M_{\text{standard}}\|_F^2$ w.r.t. increasing size of M_{standard}



Resolvent operator

Definition 1 (Resolvent) Let A be a closed operator on some Hilbert space. Let $\rho(A)$ be the set of all complex numbers μ such that $R_\mu(A) = (A - \mu I)^{-1}$ is defined and bounded.
$\rho(A)$: the resolvent set of operator A
$R_\mu(A)$: the resolvent operator of A at μ

- Given Laplace–Beltrami operator Δ
- Define $R_{a+ib}(\Delta^\gamma)$, the resolvent operator of Δ^γ at $(a + bi)$
  - (Parameters $\gamma = \frac{1}{2}, a = 0, b = 1$)
- $R_{a+ib}(\Delta^\gamma)$ is well-defined and bounded for any $(a + ib)$ not in the non-negative real line (where the spectra of Δ^γ lies in)