Structured Regularization of Functional Map Computations

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Shape Matching

- **Point-based methods**
  - [Bronstein et al. 2006],
  - [Huang et. Al 2008]...

- **Parameterization-based methods**
  - [Lipman and Funkhouser 2009]
  - [Aigerman et al. 2017]...

- **Optimal transport**
  - [Solomon et al. 2016]
  - [Mandad et al. 2017]...

- **Functional maps**
  - [Ovsjanikov et al. 2012]
  - [Ezuz and Ben–Chen 2017]...

- ...
Functional map pipeline

Eigenfunctions of Laplace–Beltrami Operator

Helmholtz equation
\[ \Delta_S f = \lambda f \]

Shape \( S \)

\[ \phi_1^S \leq \phi_2^S \leq \phi_3^S \leq \cdots \leq \phi_i^S \leq \cdots \leq \phi_k^S \]

\[ 0 = \lambda_1^S \leq \lambda_2^S \leq \lambda_3^S \leq \cdots \leq \lambda_i^S \leq \cdots \leq \lambda_k^S \]
Functional map pipeline

Function space basis

Shape $S$

$f \approx a_1 \phi_1^S + a_2 \phi_2^S + \cdots + a_k \phi_k^S = \Phi^S a$

function $f$
Functional map pipeline

Functional map definition

\( S_1 \) \( f \) \( S_2 \) \( g \)

\( \Phi^{S_1} \)

\( f \approx \Phi^{S_1} a \)

\( C a = b \)

\( \Phi^{S_2} \)

\( g \approx \Phi^{S_2} b \)

functional map: the matrix \( C \) that transports the coefficients from \( \Phi^{S_1} \) to \( \Phi^{S_2} \)
Functional map pipeline

\[ a = (\Phi^{S_1})^\dagger f \]

\[ \hat{g} = \Phi^{S_2} b \]
The functional map pipeline is illustrated in the diagram. The functional map $C$ is represented as an input image. The process is described by the equation:

$$a = (\Phi^{S_1})^T f$$

This equation maps the input $f$ to $a$. The output $a$ is then transformed to $\hat{g} = \Phi^{S_2} b$ using another functional map $\Phi^{S_2}$.
Functional map pipeline

\[ C^*_{12} = \arg\min_C \|CA - B\|_F^2 \]

\[ + w_1 \|C\Delta_1 - \Delta_2 C\|_F^2 \]

\[ + w_2 \|C\Omega_1^{\text{multi}} - \Omega_2^{\text{multi}} C\|_F^2 \]

\[ + w_3 \|C\Omega_1^{\text{orient}} - \Omega_2^{\text{orient}} C\|_F^2 \]

\[ + \ldots \]

Descriptor preservation [OBCS*12]

Laplacian commutativity [OBCS*12]

Multiplicative operators [NO17]

Orientation preservation [RPWO18]
Outline

- Laplacian commutativity – widely used
- **Drawbacks** of the standard Laplacian commutativity
  - Unbounded in the smooth setting
  - Not aligned with the ground-truth functional map
- Propose the **resolvent** Laplacian commutativity
  - Bounded operator
  - Better aligned
- Quantitative results
  - Better **stability**
  - Better **accuracy**
Reformulate the Laplacian–Commutativity term

\[ E(C) = \| C \Delta_1 - \Delta_2 C \|_F^2 \]

\[ = \| C \text{diag}(\Lambda_1) - \text{diag}(\Lambda_2) C \|_F^2 \]

\[ = \sum_{(i,j)} M_{ij} C_{ij}^2 \]

where \( M_{ij} = \left( \lambda_j^{S_1} - \lambda_i^{S_2} \right)^2 \)
Applications of the Laplacian commutativity

“Image Co–Segmentation via Consistent Functional Maps”
Fan Wang, Qixing Huang, Leonidas J. Guibas
Applications of the Laplacian commutativity

“Partial Functional Correspondence”
E. Rodolà, L. Cosmo, M.M. Bronstein, A. Torsello, D. Cremers

\[ \rho_{corr}(C) = \sum_{ij} W_{ij} C_{ij}^2 + \cdots \]
Drawbacks of the Laplacian commutativity

- Unboundedness
  - in the full LB basis (of smooth manifolds)
    \[ \| C_{12} \Delta_1 - \Delta_2 C_{12} \|^2 \to \infty \]
- Structure misalignment
Unboundedness Example

Spectrum of torus and sphere with unit area

<table>
<thead>
<tr>
<th>Size of $M$</th>
<th>$|M_{\text{standard}}|<em>F^2$ v.s. increasing size of $M</em>{\text{standard}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>50</td>
<td>0.2</td>
</tr>
<tr>
<td>100</td>
<td>0.4</td>
</tr>
<tr>
<td>150</td>
<td>0.6</td>
</tr>
<tr>
<td>200</td>
<td>0.8</td>
</tr>
</tbody>
</table>

**Torus**

**Sphere**
Unboundedness Example

$S_2: \Delta_2 = c\Delta_1$

$c \neq 1$

$S_1: \Delta_1$

$\|C_{12}\Delta_1 - \Delta_2 C_{12}\|^2 = (c - 1)^2 \|\Delta_1\|_F^2$

$\rightarrow \infty$
Structure misalignment

Mask $M_{\text{standard}}$

where $M_{ij} = (\lambda_j^{S1} - \lambda_i^{S2})^2$

Funnel-shape

$(C_{\text{ground truth}})^2$
- **Boundedness**: $\Delta \rightarrow \text{resolvent of } \Delta$
- **Structure alignment**: $\Delta \rightarrow \Delta^Y$
Subsection: Resolvent Operator

Definition

Let $A$ be a possibly unbounded linear operator (with some technical assumption), the resolvent of $A$ at $\mu$ is defined as

$$R_\mu(A) = (A - \mu I)^{-1}$$

- $\mu$ is a complex number
- $R_\mu(A)$ is defined for all $\mu$ NOT in the spectrum of $A$

$R_{a+ib}(\Delta)$ is well-defined for any $(a + ib)$ NOT in the non-negative real line (which contains the spectrum of $\Delta$)
Resolvent operator
Important tool in operator theory

- **Spectral theory**: used in the definition of spectrum
- **Unbounded self-adjoint operators**: norm-resolvent convergence

$d(A, B) = \|R_\mu(A) - R_\mu(B)\|$
Bounded resolvent Laplacian–Commutativity

Theorem 1 (Bounded Resolvent Commutativity) Let $C_{12}$ be a bounded functional map. Then in the operator norm,

$$\|C_{12}R(\Delta_1^\gamma) - R(\Delta_2^\gamma)C_{12}\|_{\text{op}}^2 < \infty$$
Bounded resolvent Laplacian-Commutativity

The graph shows the comparison of two metrics, $\|M_{\text{standard}}\|_F^2$ and $\|M_{\text{resolvent}}\|_F^2$, against the size of the matrix $M$. The x-axis represents the size of $M$, ranging from 0 to 200, and the y-axis shows the values of the metrics, ranging from 0 to 0.6. The curves illustrate the trend as the size of $M$ increases, indicating how the metrics change with respect to the size of the matrix.
Bounded resolvent Laplacian–Commutativity

- $\Delta \rightarrow$ standard Laplacian commutator
- $R_{a+ib}(\Delta^\gamma)$: well-defined and bounded
  - Introduce $\gamma$ to tune the structure of the mask
  - Our resolvent Laplacian commutator

$$E(C_{12}) = \|C_{12}A_1 - A_2 C_{12}\|_F^2 = \|C_{12} R(\Delta^\gamma_1) - R(\Delta^\gamma_2) C_{12}\|_F^2$$
Resolvent mask

- \( \Delta \) has eigenvalues \( \lambda_k \)
- \( R_i(\Delta^{1/2}) \) has eigenvalues

\[
M_{ij} = \left( \lambda_j^{S_1} - \lambda_i^{S_2} \right)^2
\]

\[
M_{ij}^{\text{Re}} = \left( \frac{\sqrt{\lambda_j^{S_1}}}{\lambda_j^{S_1} + 1} - \frac{\sqrt{\lambda_i^{S_2}}}{\lambda_i^{S_2} + 1} \right)^2
\]

\[
M_{ij}^{\text{Im}} = \left( \frac{1}{\lambda_j^{S_1} + 1} - \frac{1}{\lambda_i^{S_2} + 1} \right)^2
\]

* Def: \( R_\mu(A) = (A - \mu I)^{-1} \)
Resolvent mask

\[ \| C_{12} R(\Delta_1^\gamma) - R(\Delta_2^\gamma) C_{12} \|_F^2 = \sum_{i,j} M_{ij} C_{12}^2 \]

 Mask \( M_{\text{resolvent}} \)

...(image of resolvent mask)

where \( M_{ij} = M_{ij}^{\text{Re}} + M_{ij}^{\text{Im}} \)

...(image of funnel-shape)

Funnel-shape

...(image of mask)

\((C_{\text{ground\_truth}})^2\)
Mask reformulation of the resolvent commutativity

$$E(C_{12}) = \| C_{12} R(\Delta_1^\gamma) - R(\Delta_2^\gamma) C_{12} \|^2_F = \sum_{(i,j)} M_{ij} C_{12}^2$$

\( \gamma = 0.25 \) \hspace{1cm} \( \gamma = 0.5 \) \hspace{1cm} \( \gamma = 0.75 \) \hspace{1cm} \( \gamma = 1 \)
Penalty mask v.s. ground-truth functional map

- Standard mask
- Slanted mask
- Resolvent mask $\gamma = 0.5$
- Mean squared ground-truth

“Partial Functional Correspondences” Rodolà et al
Results: Stability (example)

Given one pair of descriptors
Compute a $k \times k$ functional map
$k^2$ variables!
Results: Stability (summary)

FAUST

per-vertex measure

Average error

$\text{standard}$  $\text{slanted}$  $\text{ours}$

$k$

$50$  $100$  $150$  $200$  $250$
Results: **Accuracy** (example)

Given one pair of descriptors
Compute a $100 \times 100$ functional map

- Standard
- Slanted
- Resolvent
- Ground-truth
Results: **Accuracy** (summary)

![Graph showing accuracy results](image)

- TOSCA

<table>
<thead>
<tr>
<th>Method</th>
<th>Accuracy (×10⁻³)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Standard (1)</td>
<td>213.0</td>
</tr>
<tr>
<td>Slanted (2)</td>
<td>190.6</td>
</tr>
<tr>
<td>Ours (3)</td>
<td>125.7</td>
</tr>
<tr>
<td>Standard + ICP</td>
<td>156.5</td>
</tr>
<tr>
<td>Slanted + ICP</td>
<td>163.2</td>
</tr>
<tr>
<td>Ours + ICP</td>
<td>90.2</td>
</tr>
<tr>
<td>Standard + BCICP</td>
<td>81.8</td>
</tr>
<tr>
<td>Slanted + BCICP</td>
<td>125.3</td>
</tr>
<tr>
<td>Ours + BCICP</td>
<td>61.5</td>
</tr>
</tbody>
</table>

Geodesic Error ($\times 10^{-3}$)
Results: **Correlation** (fMap penalty v.s. pMap accuracy)

![Graph showing correlation between fMap penalty and pMap accuracy](image)

- **Mask penalty** vs. **Average geodesic error (direct measure)**

Legend:
- **Standard**
- **Slanted**
- **Ours**
Results: **Stability under remeshing and coarsening**

Source

- $n_v = 6890$
- $n_v = 200$
- $n_v = 300$
- $n_v = 500$
- $n_v = 1000$
- $n_v = 3000$
- $n_v = 5000$
- $n_v = 6890$

Target

- Solve for $100 \times 100$

Standard Mask

Complex Mask

Ours
Summary

• Shape matching – functional map pipeline
• Laplacian commutativity – widely used
• **Drawbacks** of the standard Laplacian commutativity
  • Unbounded in the smooth setting
  • Not aligned with the ground-truth functional map
• Propose the **resolvent** Laplacian commutativity
  • Bounded operator
  • Aligned with the funnel shape
• Results
  • Better accuracy
  • Better stability
Thanks for your attention

Structured Regularization of Functional Map Computations

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Lemma 2. Let $\Delta_1$ and $\Delta_2$ be Laplacians on compact, connected, oriented surfaces $M_1$ and $M_2$, respectively. Let $C_{12}: L_2(M_1) \to L_2(M_2)$ be a bounded operator. If $\gamma > \frac{1}{2}$, then:

$$\|C_{12} R_\mu (\Delta_1^\gamma) - R_\mu (\Delta_2^\gamma) C_{12}\|_{HS}^2 < \infty$$

Where $\mu$ is any complex number not on the non-negative real line.
Reformulate the Laplacian–Commutativity term

\[ E(C_{12}) = \|C_{12}\Delta_1 - \Delta_2 C_{12}\|_F^2 \]

\[ = \|C_{12}\text{diag}(\Lambda_1) - \text{diag}(\Lambda_2)C_{12}\|_F^2 \]

\[ = \|C_{12} \otimes (1_{k_2} \Lambda_1^T) - (\Lambda_2 1_{k_1}^T) \otimes C_{12}\|_F^2 \]

\[ = \|((1_{k_2} \Lambda_1^T - \Lambda_2 1_{k_1}^T) \otimes C_{12}\|_F^2 \]

\[ = \sum_{(i,j)} M \otimes (C_{12})^2 \]

Note: \( \otimes \) is the entry-wise matrix multiplication.
Results: **Stability (summary)**

**FAUST**

- **per-vertex measure**

- **direct measure**

![Graphs showing average error vs. k for FAUST with different measures and initializations.](image-url)
Given one pair of descriptors, compute a $100 \times 100$ functional map. Corresponding point-wise map.

Results: **Accuracy** (example)
Unboundedness Example

Unit Sphere  Unit Torus

\[
\lambda_k = \begin{cases} 
Torus & k = 0, 20, 40, 60, 80, 100 \\
Sphere & k = 500 \\
Weyl Estimate & k = 1, 000 
\end{cases}
\]
Unbounded standard Laplacian–Commutativity

\[ \|M_{\text{standard}}\|_F^2 \text{ w.r.t. increasing size of } M_{\text{standard}} \]
Definition 1 (Resolvent) Let $A$ be a closed operator on some Hilbert space. Let $\rho(A)$ be the set of all complex numbers $\mu$ such that $R_\mu(A) = (A - \mu I)^{-1}$ is defined and bounded. 

$\rho(A)$: the resolvent set of operator $A$

$R_\mu(A)$: the resolvent operator of $A$ at $\mu$

- Given Laplace–Beltrami operator $\Delta$
- Define $R_{a+ib}(\Delta^\gamma)$, the resolvent operator of $\Delta^\gamma$ at $(a + bi)$
  - (Parameters $\gamma = \frac{1}{2}, a = 0, b = 1$)
  - $R_{a+ib}(\Delta^\gamma)$ is well-defined and bounded for any $(a + ib)$ not in the non-negative real line (where the spectra of $\Delta^\gamma$ lies in)