Continuous and Orientation-preserving Correspondences via Functional Maps

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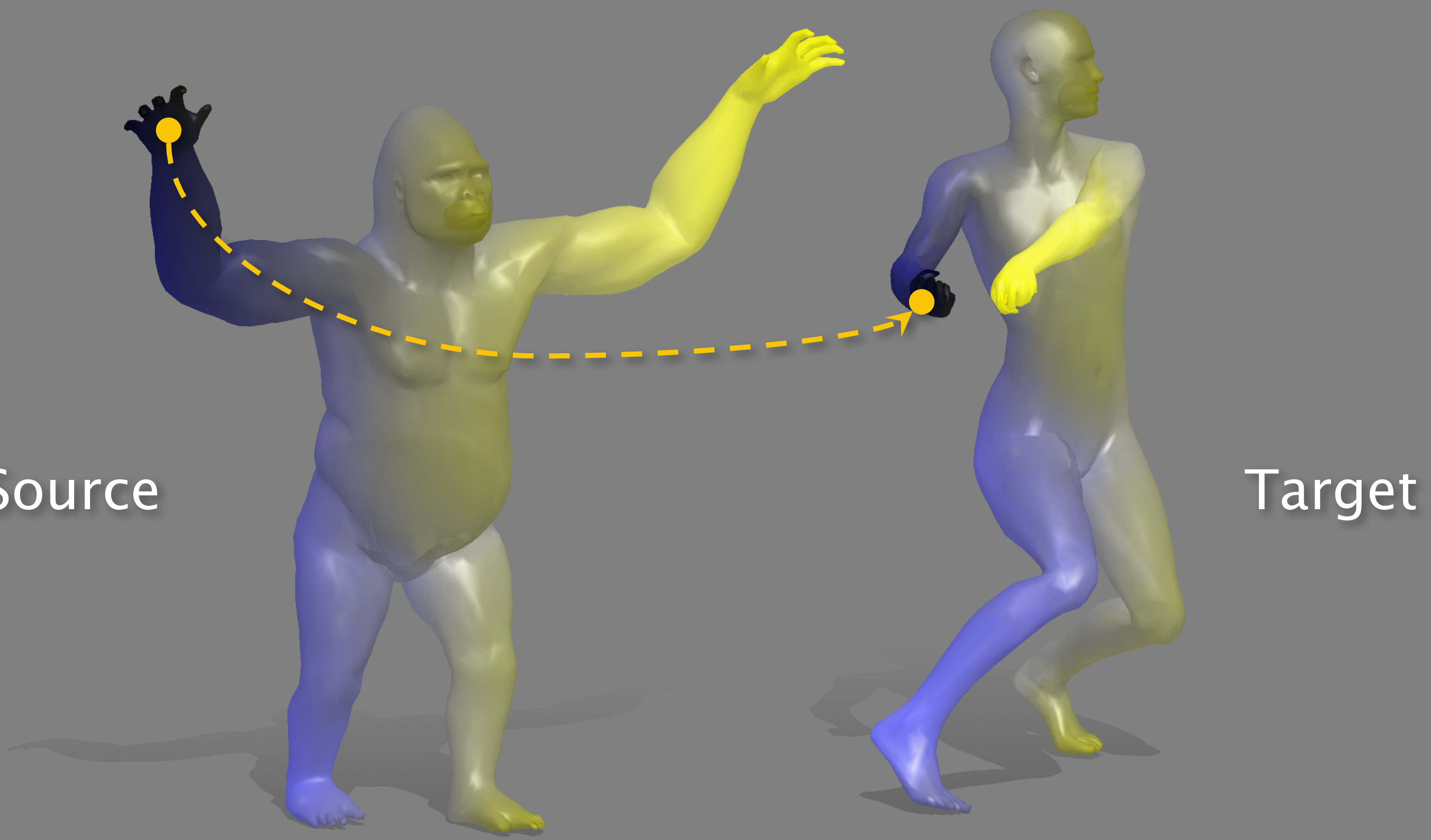
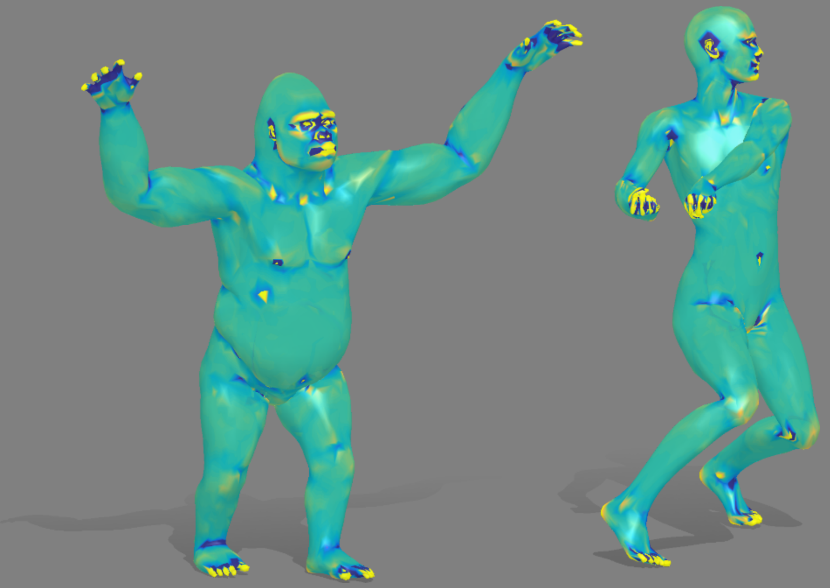
Shape matching



Corresponding descriptors

Gaussian curvatures

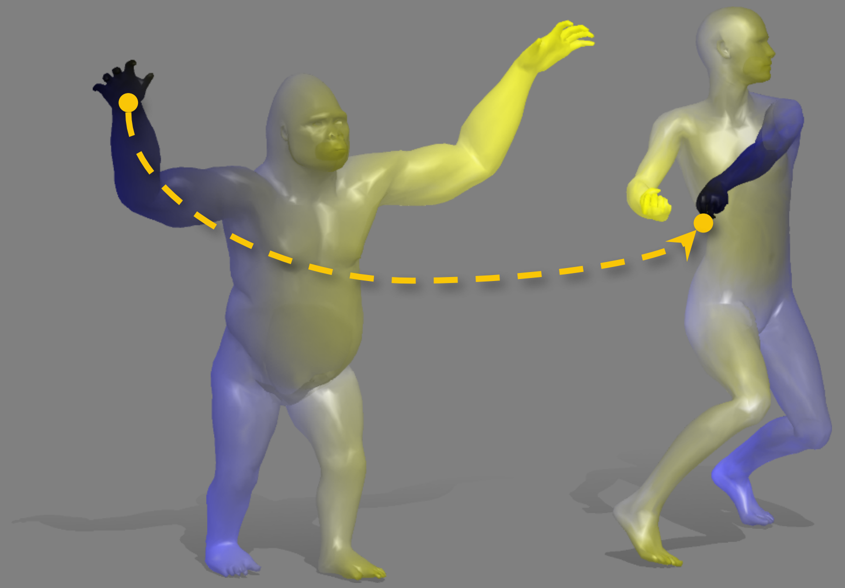


Joint segmentation

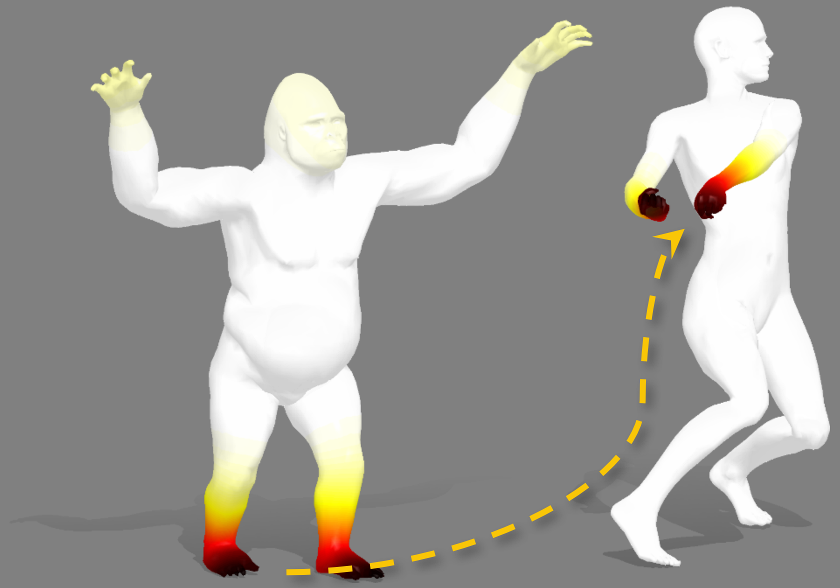
Wave kernel signature

Problems

Left-right ambiguity



Leg-arm ambiguity



How to break the symmetry ambiguity?

Our solution: define orientation-preserving operators in the functional map framework.

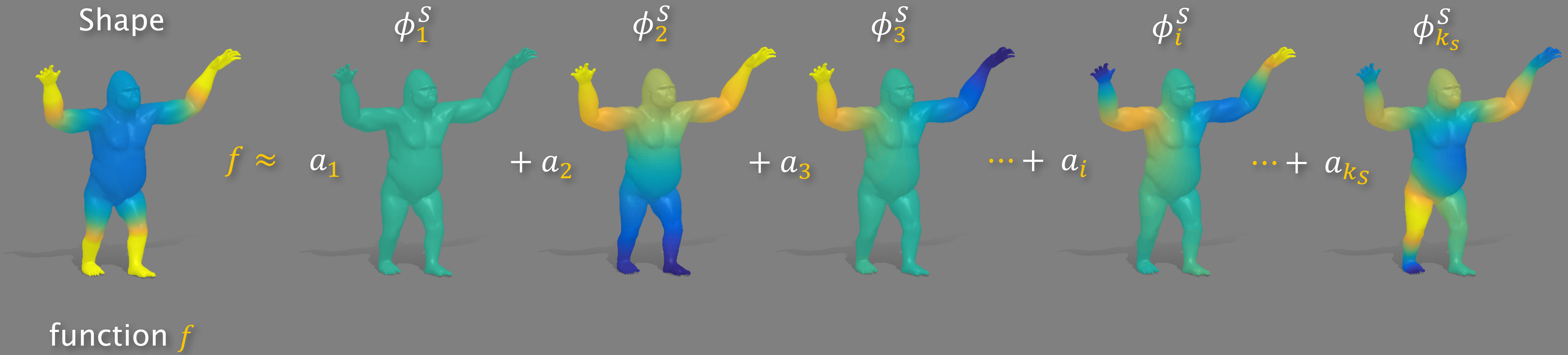
Q1: what is the standard functional map pipeline to solve the shape matching problem?

Q2: how to define the orientation-preserving operator? Why does it work?

Q1: Functional map pipeline

Laplacian–Beltrami operator

Q1: Functional map pipeline — Laplacian–Beltrami operator



Q1: Functional map pipeline

Source

$f \approx \Phi^S a$

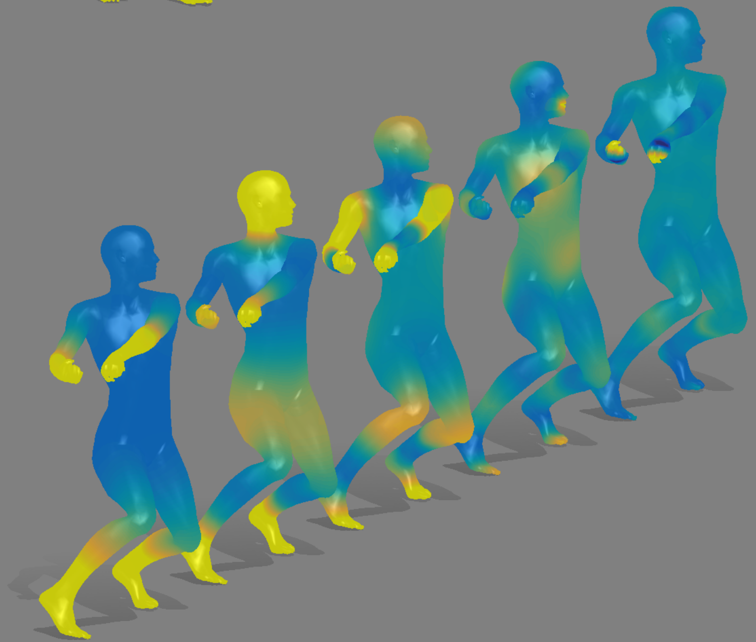
$$\{f_i\}_{i=1}^k \xrightarrow{\Phi^S} \{a_i\}_{i=1}^k$$

$$C a_i = b_i \quad \forall i = 1, \cdots k$$

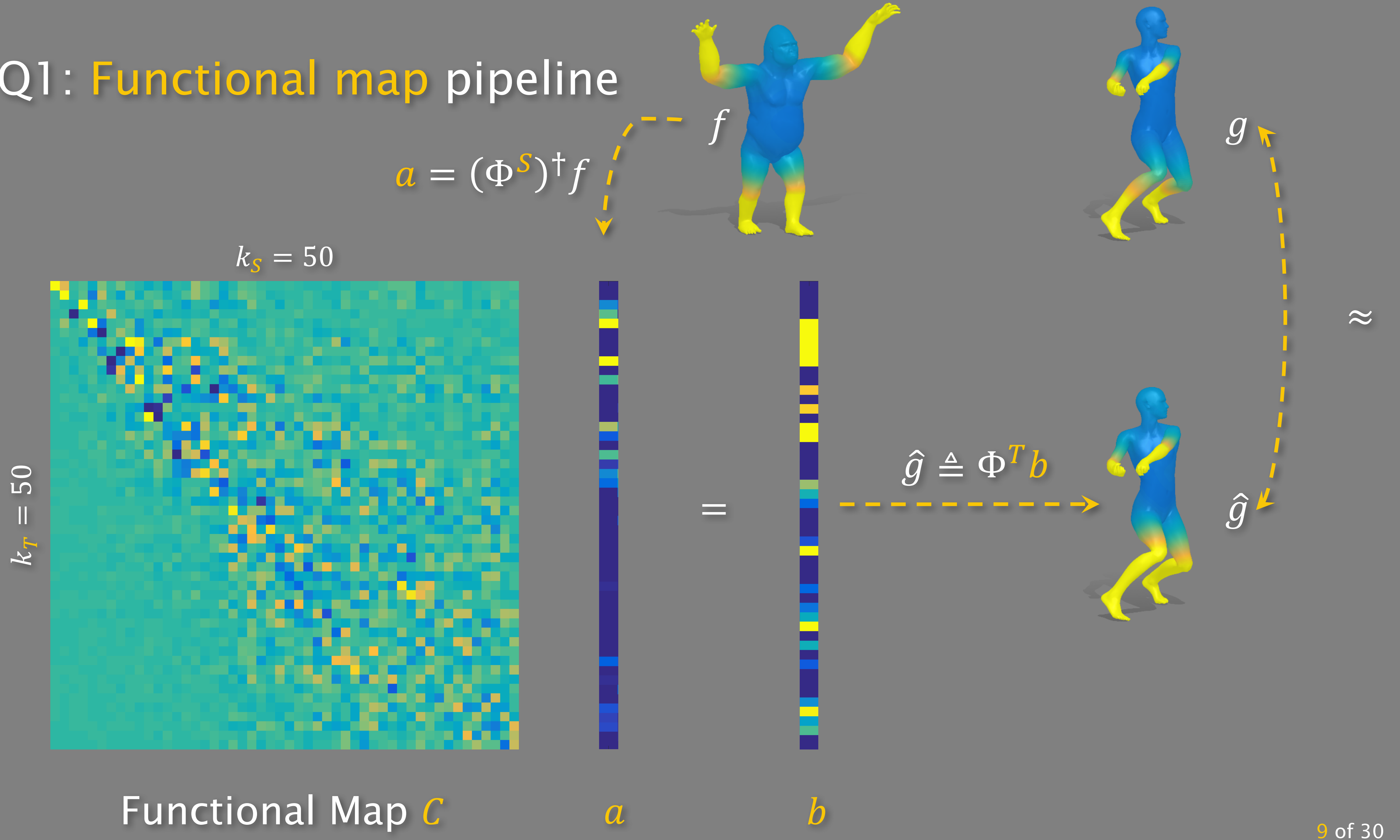
$$\{g_i\}_{i=1}^k \xrightarrow{\Phi^T} \{b_i\}_{i=1}^k$$

Target

$g \approx \Phi^T b$



Q1: Functional map pipeline



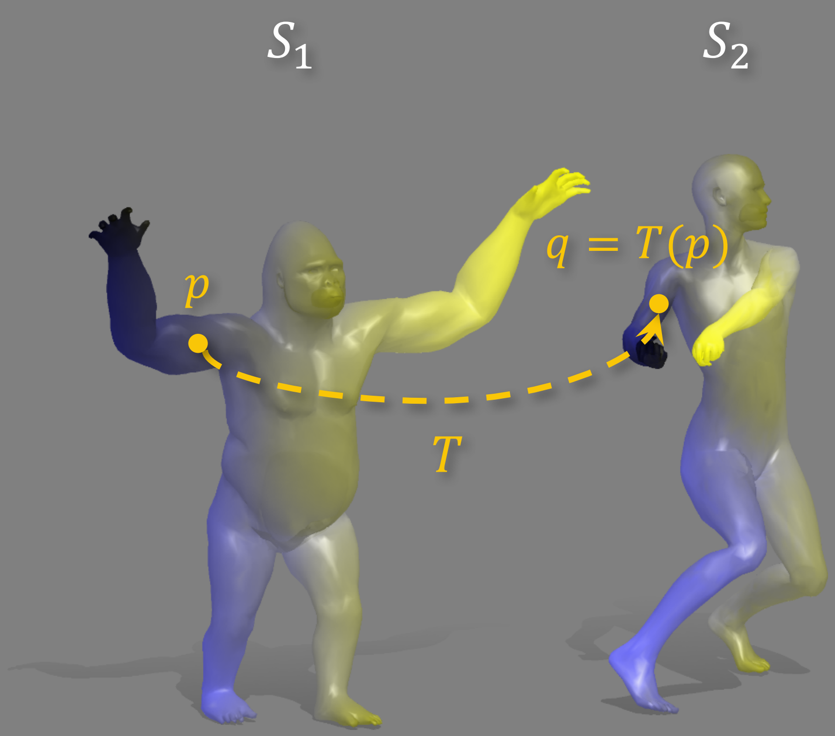
Q1: Functional map pipeline

How to find C ?

Given a pair of shapes S_1, S_2, with Laplacian operators $\Delta_{S_1}, \Delta_{S_2}$

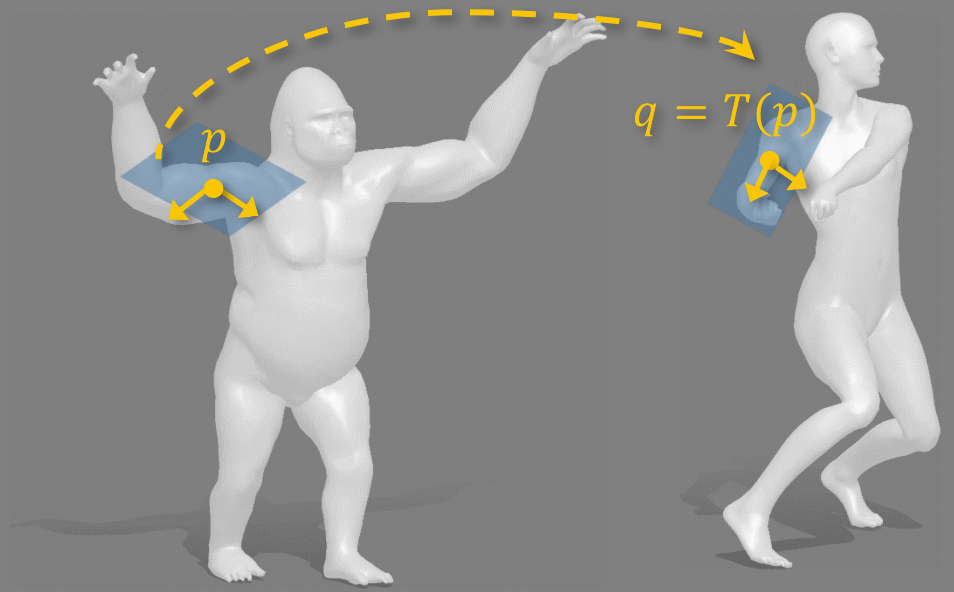
- Compute the functional bases on the two shapes. Store them in matrices Φ^{S_1}, Φ^{S_2}
- Project the corresponding descriptors $\{f_i, g_i\}_{i=1}^{k}$ into the functional space. Store the coefficients in matrices $A = [a_1 \cdots a_k], B = [b_1 \cdots b_k]$
- Solve $C^* = \underset{C \in R^{k_T \times k_S}}{\operatorname{argmin}} E(C) = \|CA - B\|^2 + \left\|C\Delta_{S_1} - \Delta_{S_2} C\right\|^2$
- Orientation-preservation term $\sum_{i=1}^{k} \left\|C\Omega_{f_i} - \Omega_{g_i} C\right\|^2$

Q2: Orientation-preserving operators



Map differential

$dT: \mathcal{T}_p(S_1) \to \mathcal{T}_q(S_2)$



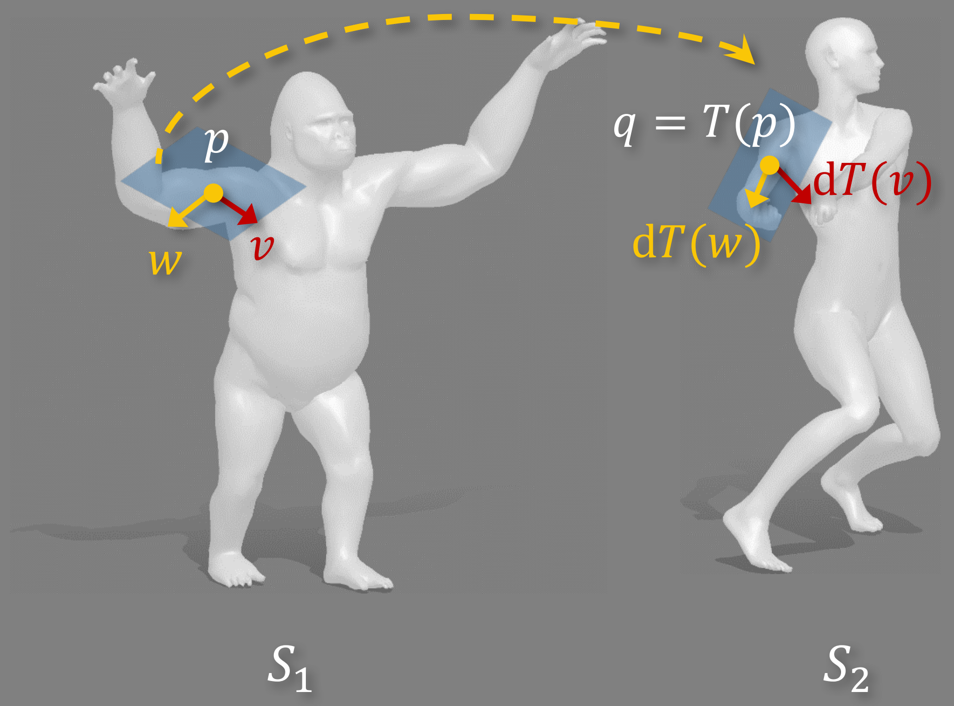
$\mathcal{T}_x(S_i)$: tangent space at $x \in S_i$

A smooth map T is orientation preserving if and only if dT is orientation preserving

Q2: Orientation-preserving operators

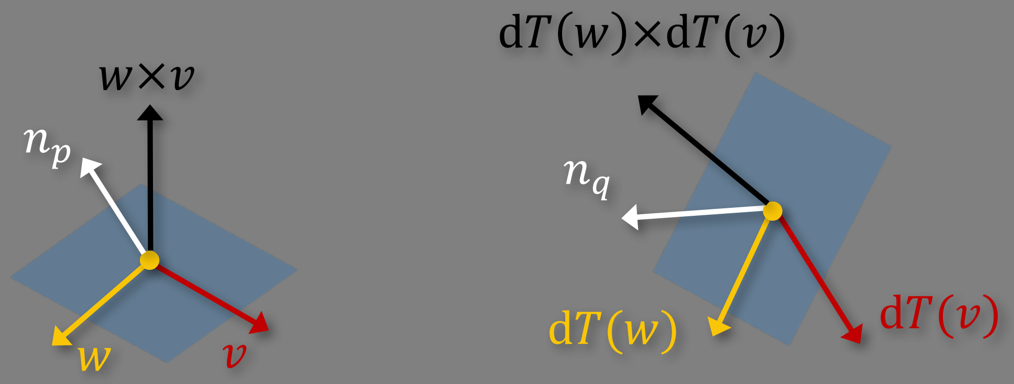
Map differential

$dT: \mathcal{T}_p(S_1) \to \mathcal{T}_q(S_2)$



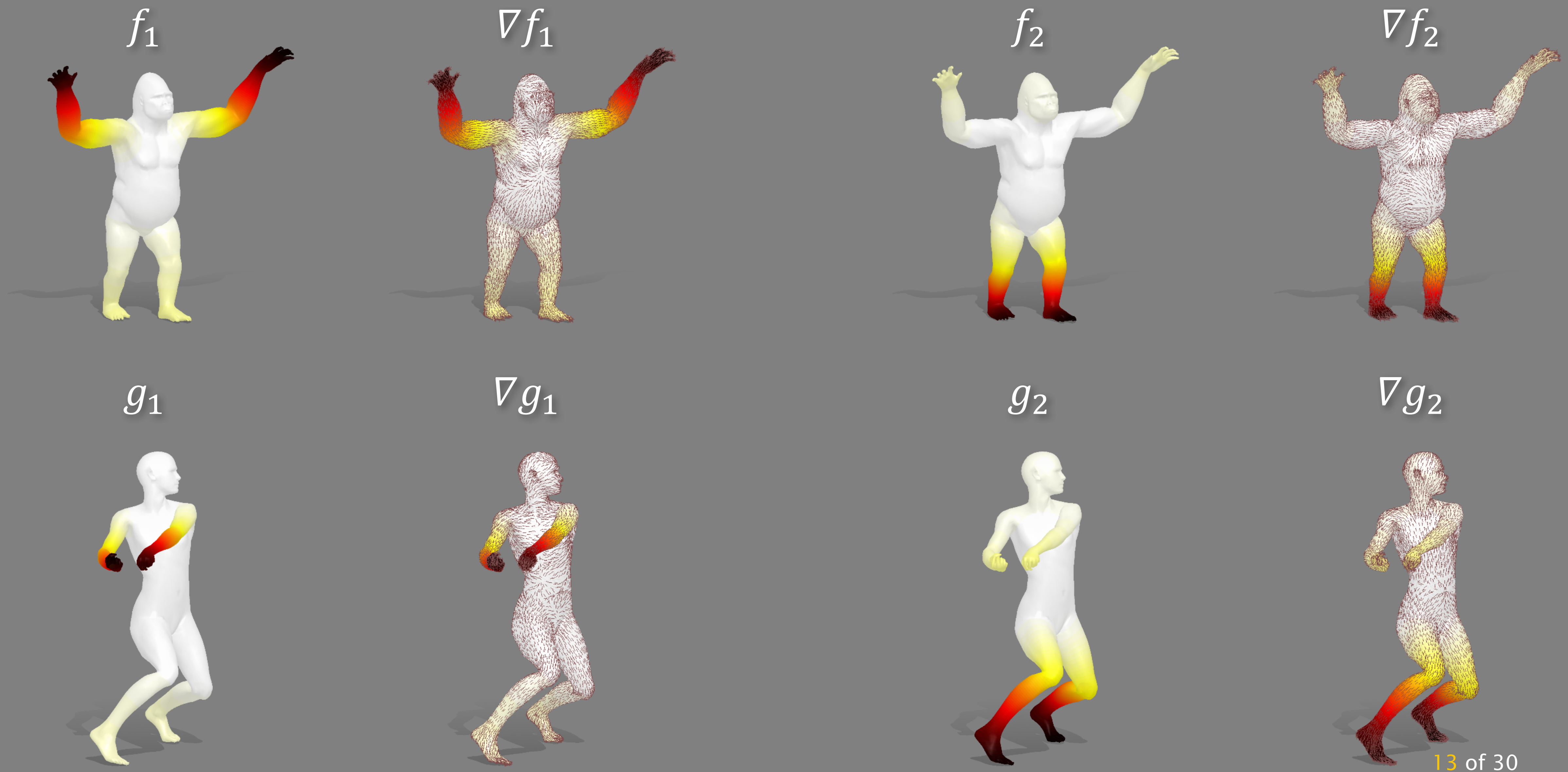
Frame at p: (w, v, n_p)

Frame at $q = T(p)$: $\left(dT(w), dT(v), n_{T(p)}\right)$

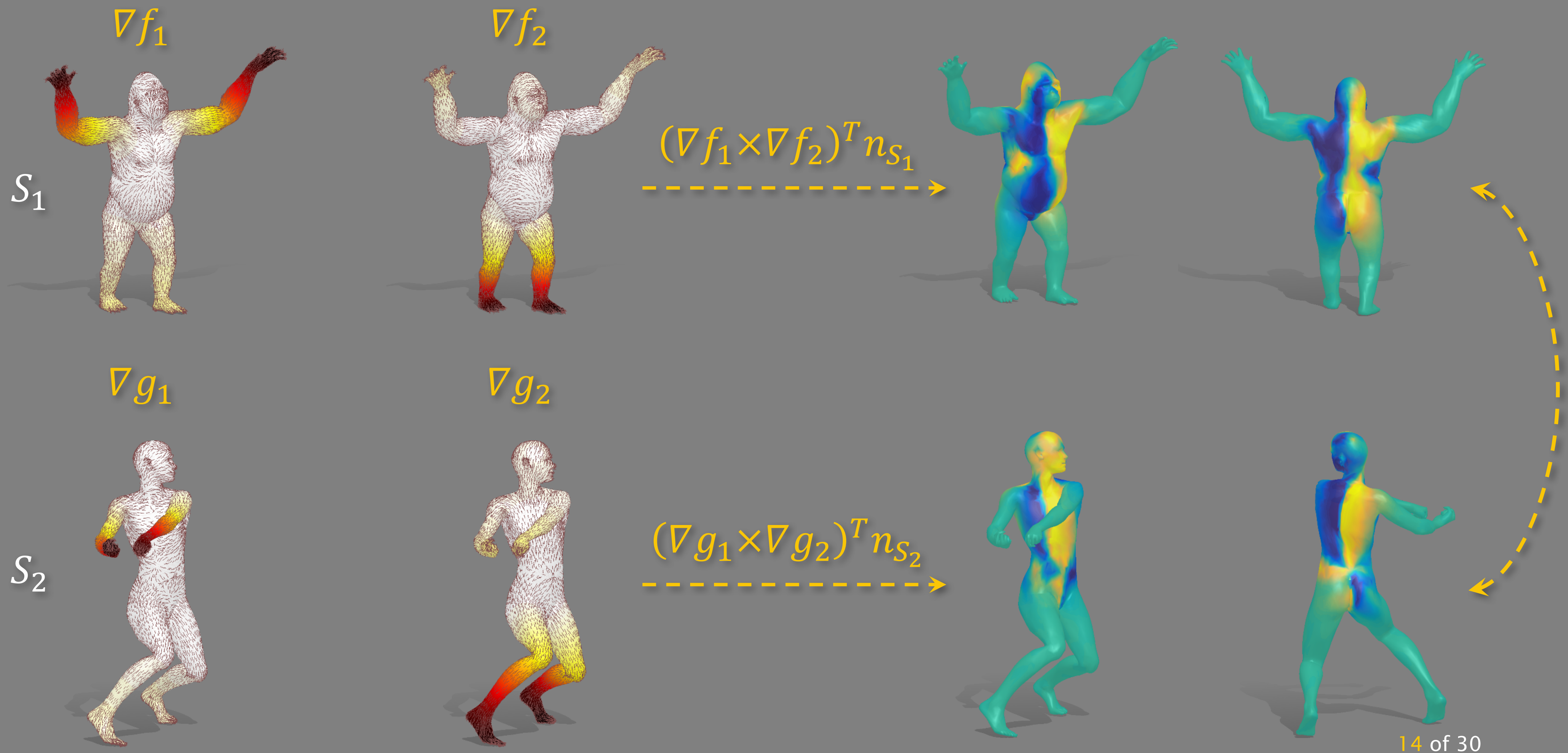


$(w \times v)^T n_p$ should have the same sign as $\left(dT(w) \times dT(v)\right)^T n_{T(p)}$

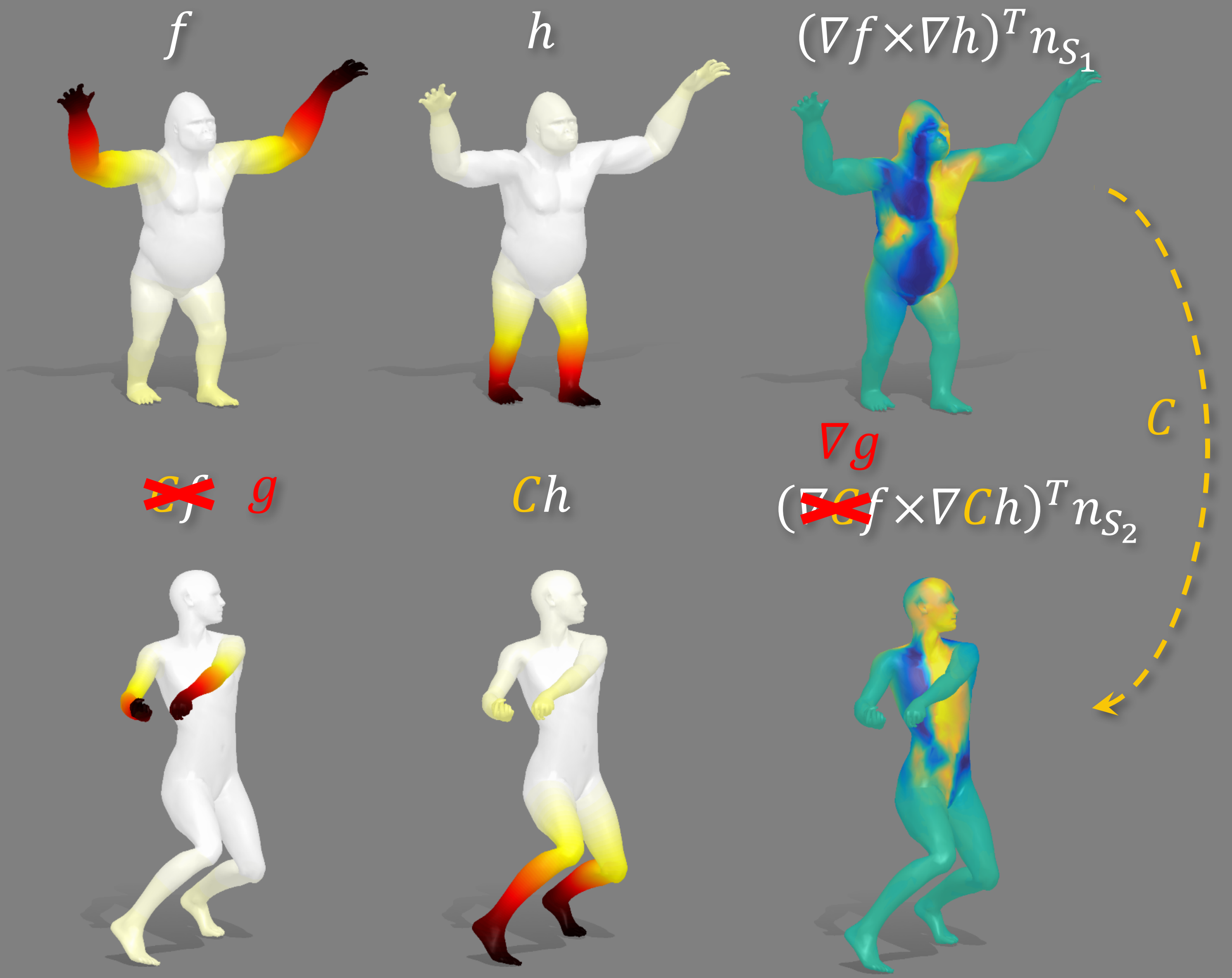
Q2: Orientation-preserving operators



Q2: Orientation-preserving operators



Q2: Orientation-preserving operators



non-linear

linear! – $\Omega(\cdot)$

Q2: Orientation-preserving operators

Shape

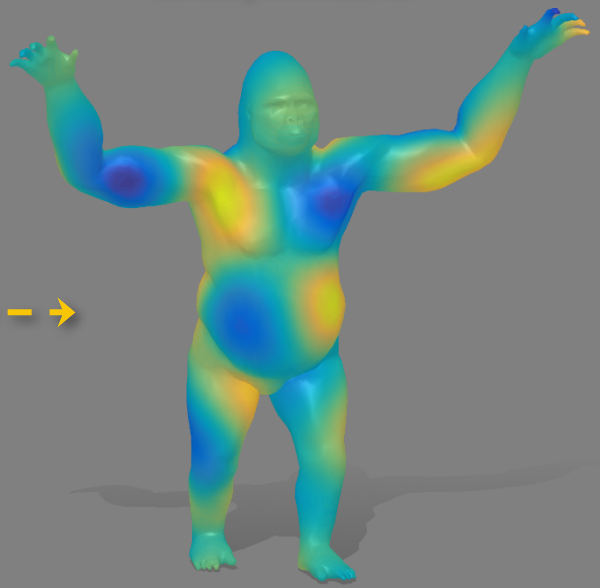
A descriptor function f

With symmetry ambiguity

$\Omega(\cdot)$

$\Omega(f)$

antisymmetric



Q2: Orientation-preserving operators

Shape

A descriptor function f
With symmetry ambiguity

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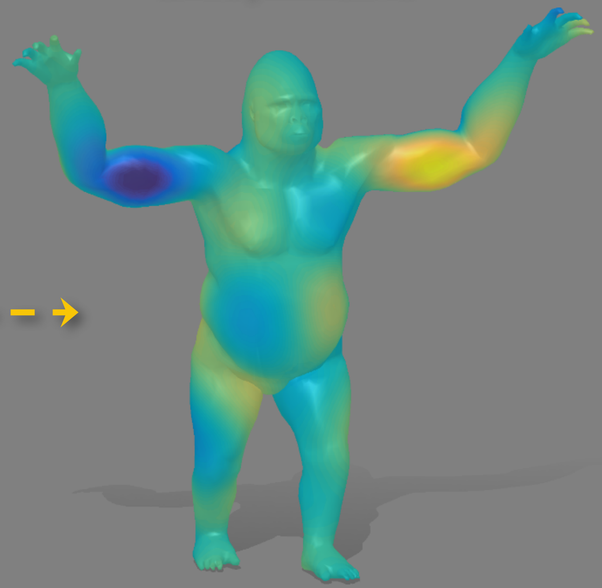


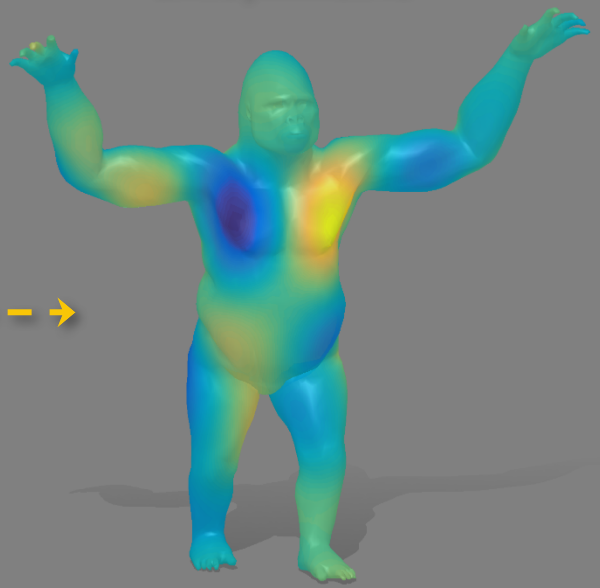
Q2: Orientation-preserving operators

Shape

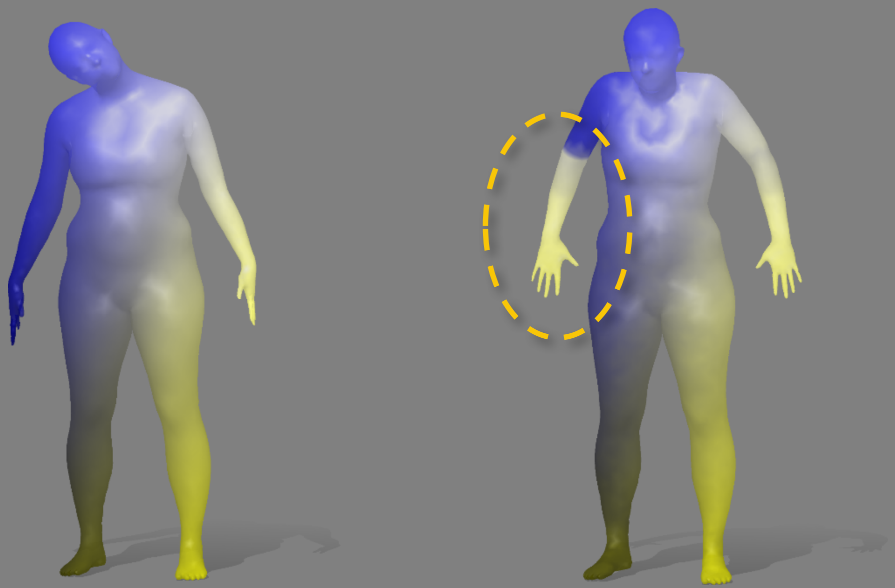
A descriptor function f
With symmetry ambiguity

$\Omega(\cdot)$

$\Omega(f)$
antisymmetric



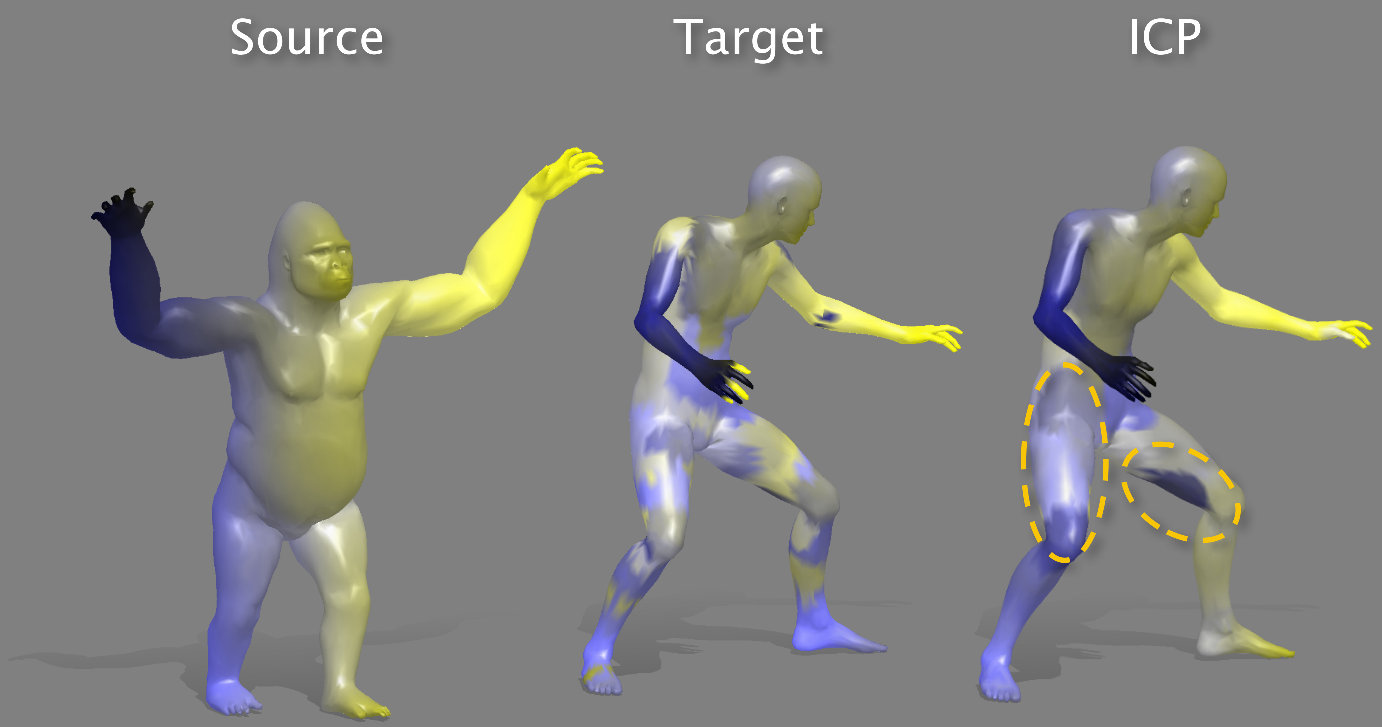
Orientation-preserving operators – defects



Intrinsic – no guarantee

Induced – depends on the descriptors

Refinement! – iterative closest point (ICP)?



Problems of ICP:
- discontinuous
- Only in the spectral domain – no spatial info

A better refinement to improve the quality?

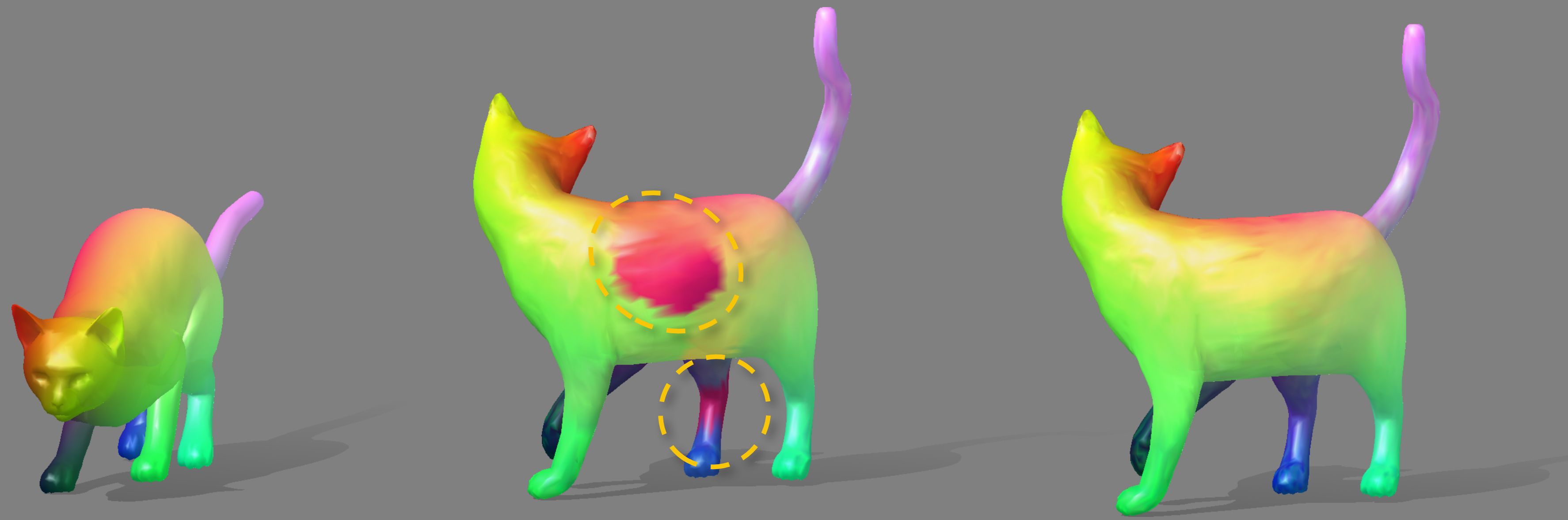
Without ground-truth correspondences,
how do we measure the quality of a map?

Refinement! – improve smoothness

smoother – better!

Refinement! – remove outliers

Source | Target: map 01 | Target: map 02



smoother (no outliers)
– better!

Refinement! – improve coverage

Source | Target: map 01 | Source | Target: map 02

Black region: not covered (no correspondence)

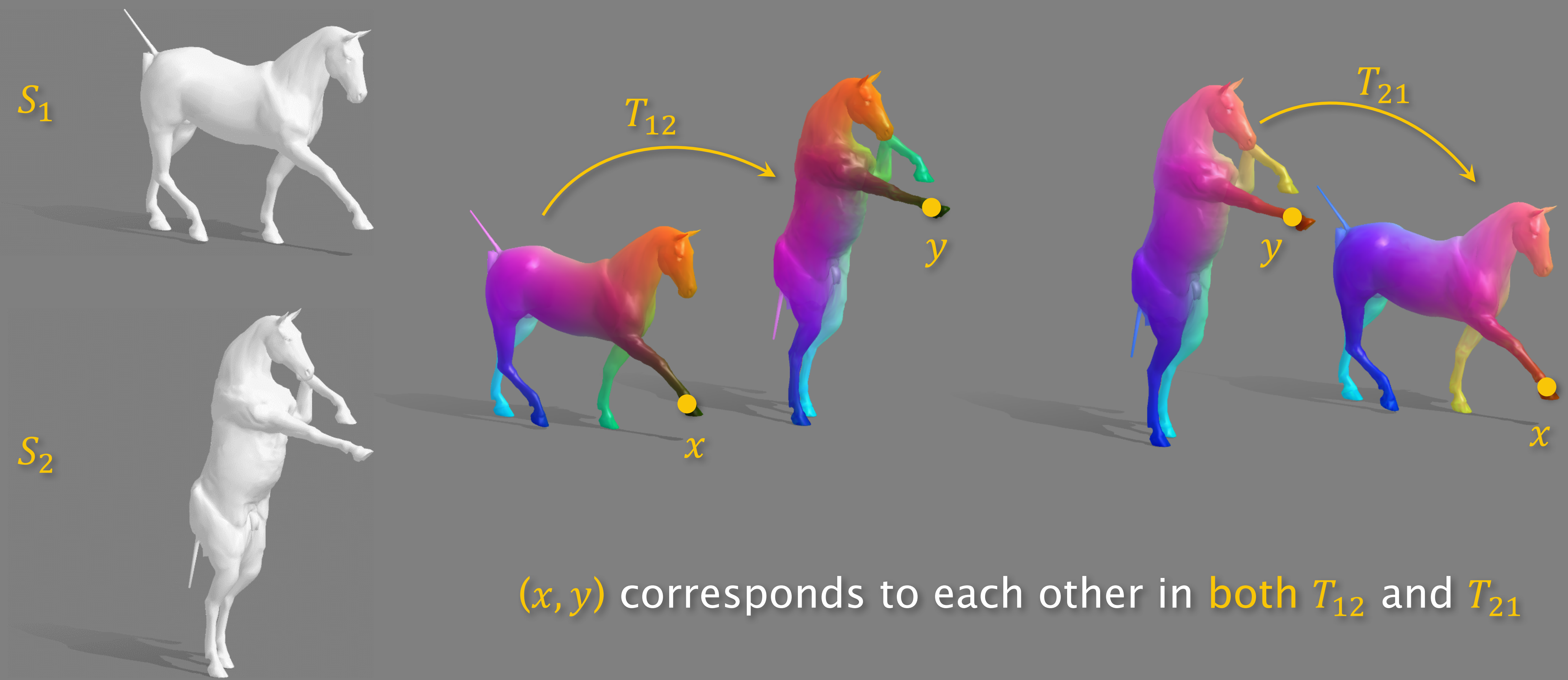
Coverage: 48.9%

Coverage: 81.4%

Coverage: #vertex (or surface area) % covered by the map

More vertices covered – better!

Refinement! – improve bijectivity



(x, y) corresponds to each other in both T_{12} and T_{21}

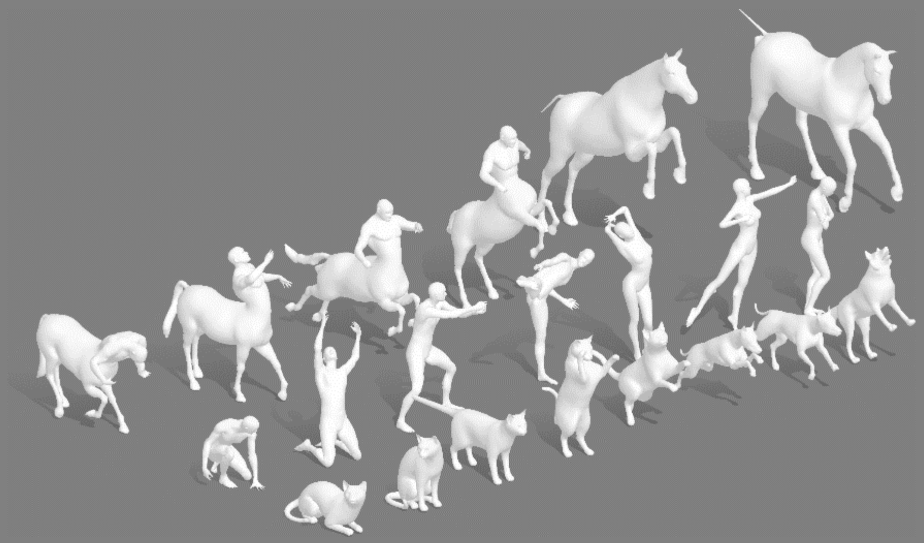
Bijective and Continuous ICP (BCICP)

code

- Bijectivity
  - Soft constraints: $T_{21} \circ T_{12}$ and $T_{12} \circ T_{21}$ are close to identity
- Continuity
  - Smooth the displacement vector field
- Coverage
  - Find the nearest neighbor with the largest preimage size
- Outliers
  - Detected by edge distortion and fixed by the nearest neighbor classified as an inlier

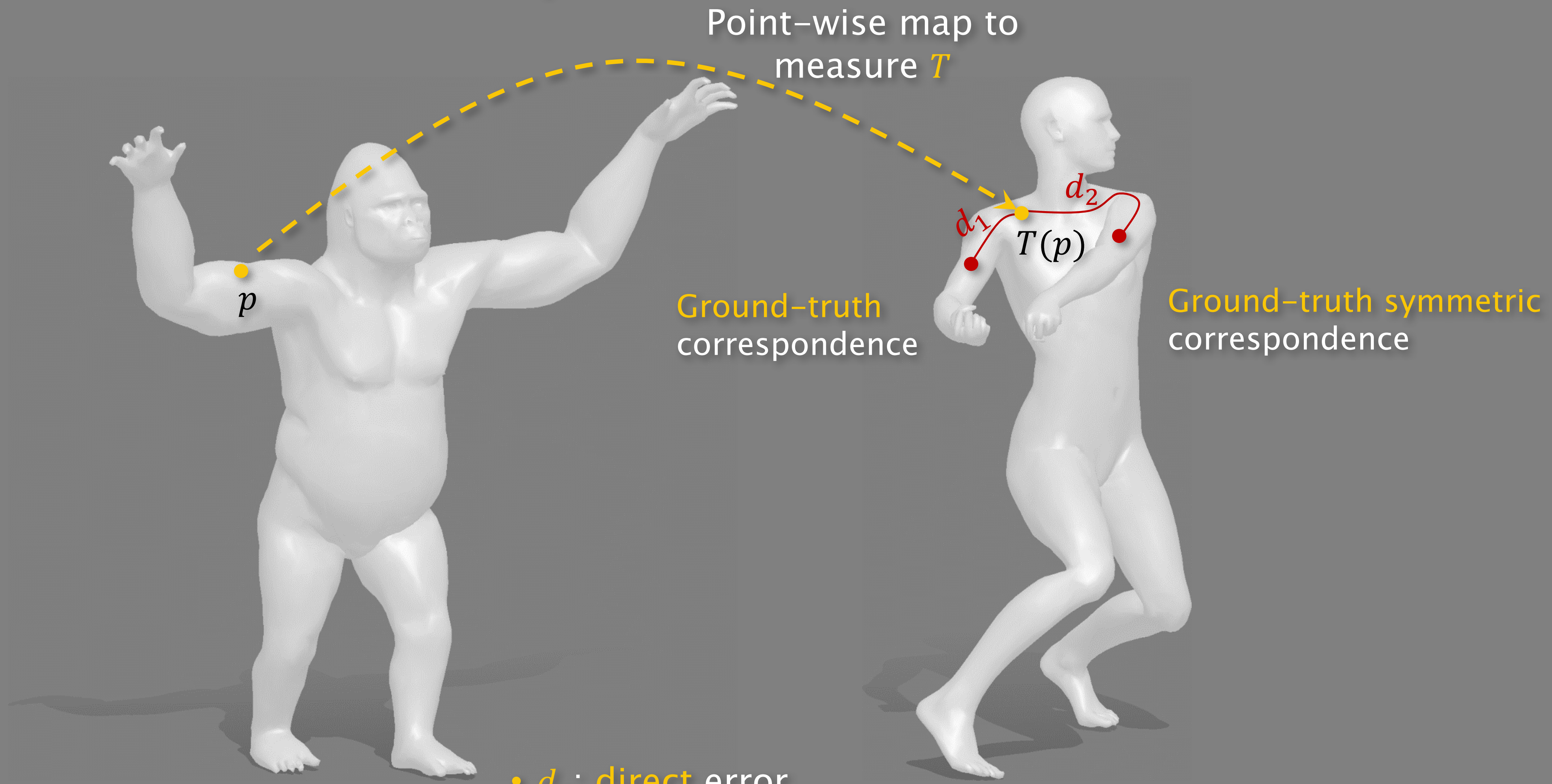
Benchmark datasets

FAUST dataset [Bogo et al. 2014]
- 10 different humans in 10 different poses
- We tested on
  - 200 isometric pairs
  - 400 non-isometric pairs



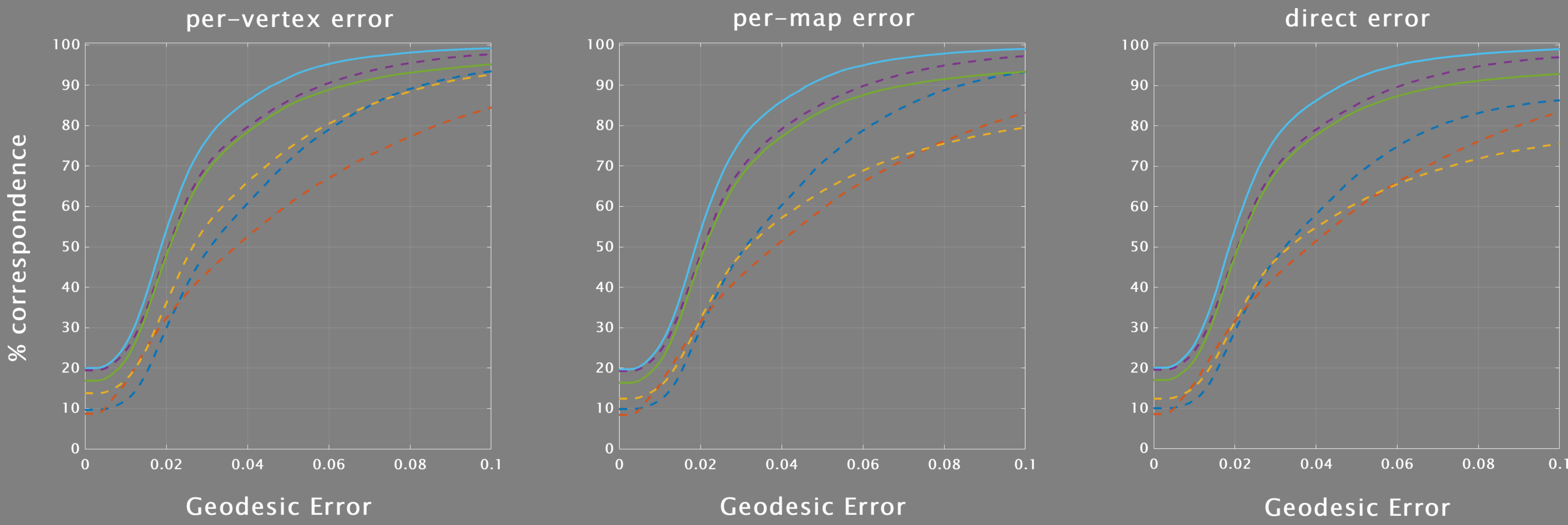
TOSCA dataset [Bronstein et al 2008]
- 80 different humans and animals in 9 categories
- We tested on
  - 568 isometric pairs
  - 190 non-isometric pairs

Measurement – Accuracy



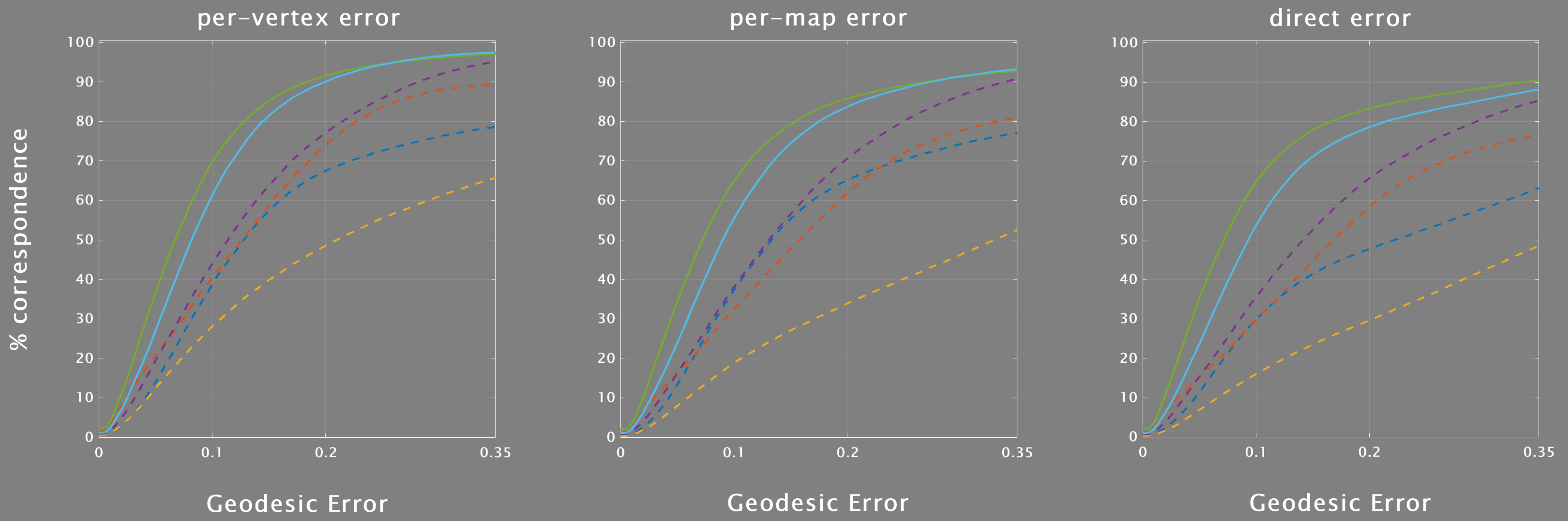
- d_1: direct error
- d_2: symmetric error
- $\min(d_1, d_2)$: per-vertex error

Results – FAUST Isometric dataset (200 pairs)



Solid lines: our methods (with different descriptors)
Dashed lines: state-of-the-art methods

Results – TOSCA non-Isometric dataset (190 pairs)



Solid lines: our methods (with different descriptors)
Dashed lines: state-of-the-art methods

Summary

- Introduce orientation-preserving operator into functional map framework
- Propose a refinement technique, BCICP, which improves the bijectivity, continuity, and coverage
- Verify the usefulness of the orientation-preserving operator and the BCICP refinement on large datasets, w.r.t. different measurements

Thanks for your attention ☺

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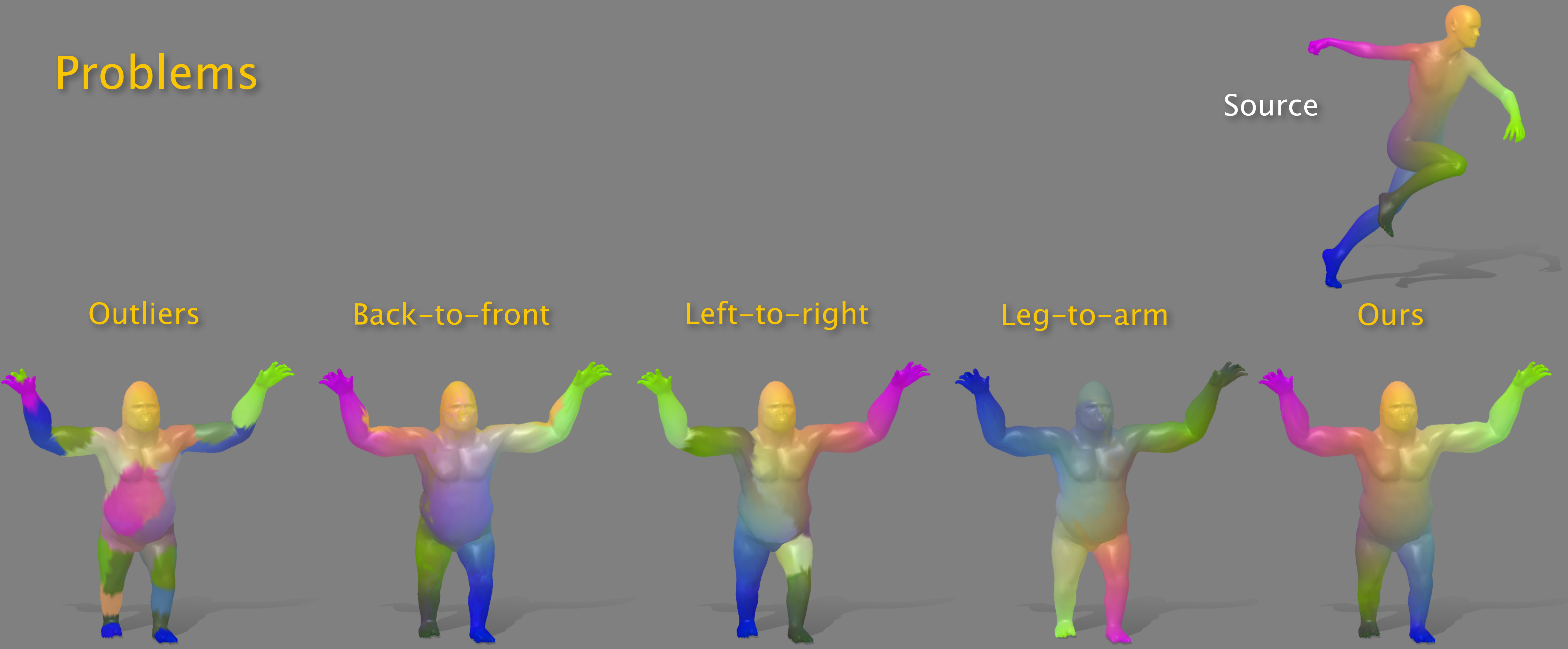
Supplementary

Problems



Results – Summary

Dataset (#pairs) / Measurement	FAUST		TOSCA	
	Isometric 200	Non-isometric 400	Isometric 586	Non-isometric 190
Per-vertex	17.8%	24.1%	31.4%	38.3%
Per-map	17.5%	18.4%	14.6%	37.1%
Direct	17.5%	18.4%	38.8%	43.5%

Relative improvement of our method (directOp + BCICP)
over the best baseline methods
w.r.t. the average error of three measurements

Q2: Orientation-preserving operators

Encode into functional map framework:

- Given a functional map C maps function on S_1 to function on S_2
- For every pair of corresponding descriptor $(f, g), f \in \mathcal{F}(S_1), g \in \mathcal{F}(S_2)$
- We can add the orientation-preserving constrain $C\left((\nabla f \times \nabla h)^T n_{S_1}\right) = \left(\nabla g \times \nabla C(h)\right)^T n_{S_2}$
  - $\nabla f, \nabla g$: vector field on the source and target mesh resp.
  - $h, C(h)$: functions defined on the source and target resp.
  - $\nabla h, \nabla C(h)$: vector field
  - $(\nabla f \times \nabla h)^T n_{S_1}$ defines a function on the source
  - $\left(\nabla g \times \nabla C(h)\right)^T n_{S_2}$ defines a function on the target
  - they should correspond to each other! – use functional map to transport the function
- $C\left((\nabla f \times \nabla h)^T n_{S_1}\right) = \left(\nabla g \times \nabla C(h)\right)^T n_{S_2}$
- $\text{OrientationPreserving}(C)$ for every pair of corresponding descriptors (f, g)

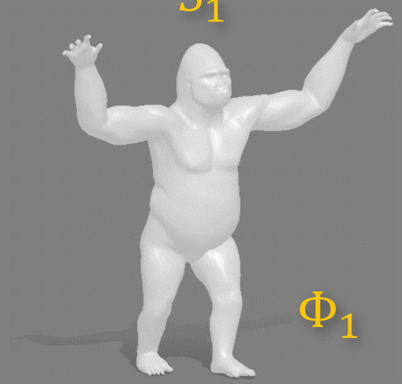
Q2: Orientation-preserving operators

- Define $\omega(w, v, n) = (w \times v, n)$
- If T is orientation preserving, we have $\text{sign}\left(\omega(w, v, n_p)\right) = \text{sign}\left(\omega(\mathrm{d}T(w), \mathrm{d}T(v), n_{T(p)})\right)$ for any pair of $w, v \in \mathcal{T}_p(S_1)$, and for any vertex p

Encode into functional map framework:

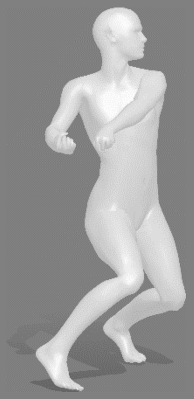
- Given a pair of corresponding descriptor $(f, g), f \in \mathcal{F}(S_1), g \in \mathcal{F}(S_2)$
- $\forall\ h \in \mathcal{F}(S_1)$, it is mapped to $C(h) \in \mathcal{F}(S_2)$ via a functional map C
- Orientation-preserving:
  - For any point $p \in S_1$ (corresponds to $q \in S_2$ via C)
  - $\omega(\nabla f(p), \nabla h(p), n_p) \approx \omega\left(\nabla g(q), \nabla\left(C(h)\right)(q), n_q\right)$
  - Define a function $\Omega(\cdot,\cdot,\cdot) \in \mathcal{F}(S_1)$ such that $\Omega(p) = \omega(\cdot_p, \cdot_p, \cdot_p)$
  - $C\left(\Omega(\nabla f, \nabla h, n_{S_1})\right) = \Omega(\nabla g, \nabla C(h), n_{S_2})$
- $\text{OrientationPreserving}(C)$ for every pair of corresponding descriptors (f, g)

BCICP refinement – improve bijectivity



Notation:

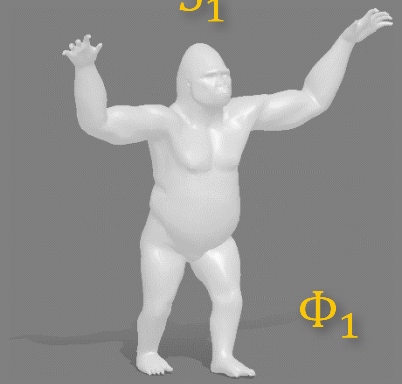
- Point-wise map (vector!) T_{12}: i–th vertex on S_1 is mapped to $T_{12}(i)$–th vertex on S_2
- Shape S_i has Laplacian–Beltrami Basis Φ_i
- Functional map (matrix!) C_{12}: maps the functional space $\mathcal{F}(S_1|\Phi_1)$ to $\mathcal{F}(S_2|\Phi_2)$

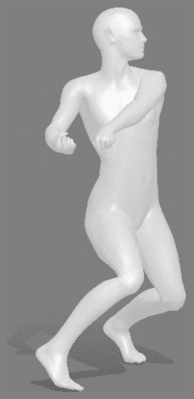


Recall ICP:

- C_{12} is associated with T_{21}
  - Arbitrary function $f \in \mathcal{F}(S_1)$ can be transported to S_2 in two ways:
    - Using the point–wise map directly, i.e., $g = f(T_{21}) \in \mathcal{F}(S_2)$
    - Use the functional map, i.e., $g = \Phi_2\left(C_{12}\left(\Phi_1^{\dagger} f\right)\right)$
- Two transportations should give similar result – for any f
- Minimize $\|\Phi_2 C_{12} - \Phi_1(T_{21},:)\|^2$
  - ICP: alternatively solve for C_{12} and T_{21}

BCICP refinement – improve bijectivity





Bijective in the spectral domain:

- $\|\Phi_2 C_{12} - \Phi_1(T_{21},:)\|^2 + \|\Phi_2 C_{21} - \Phi_1(T_{12},:)\|^2$

Bijective in the spatial domain:

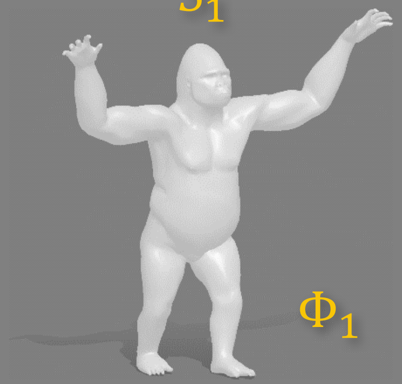
- $T_{21} \circ T_{12}$: maps S_1 to itself (S_1)
- We can add a similar term $\|\Phi_1 C_{11} - \Phi_1(T_{21} \circ T_{12},:)\|^2$
  - where C_{11} is an auxiliary variable, maps the functional space $\mathcal{F}(S_1|\Phi_1)$ to itself
- We can similarly define the energy for $T_{12} \circ T_{21}$

New energy

$$\begin{aligned} &\lambda_1\|\Phi_2 C_{12} - \Phi_1(T_{21},:)\|^2 \\ &+ \lambda_2\|\Phi_2 C_{21} - \Phi_1(T_{12},:)\|^2 \\ &+ \lambda_3\|\Phi_1 C_{11} - \Phi_1(T_{21} \circ T_{12},:)\|^2 \\ &+ \lambda_4\|\Phi_2 C_{22} - \Phi_2(T_{12} \circ T_{21},:)\|^2 \end{aligned}$$

BCICP refinement – improve bijectivity

S_1

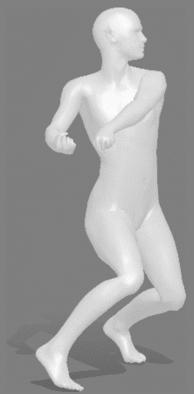


Φ_1

ICP energy

$$\|\Phi_2 C_{12} - \Phi_1(T_{21},:)\|^2$$

S_2

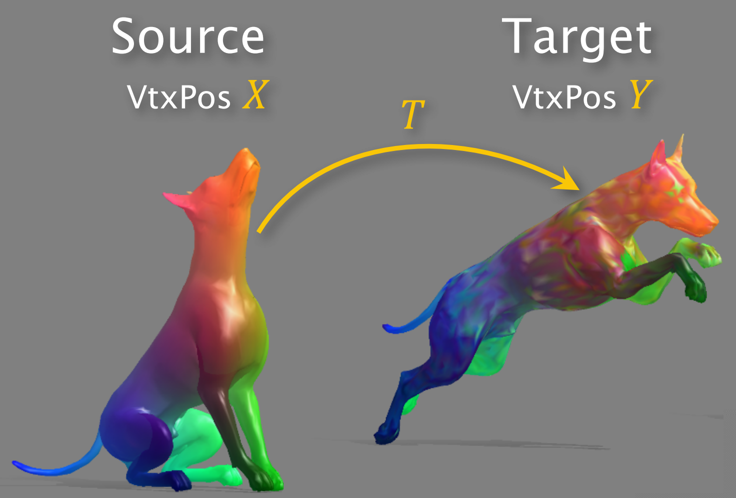


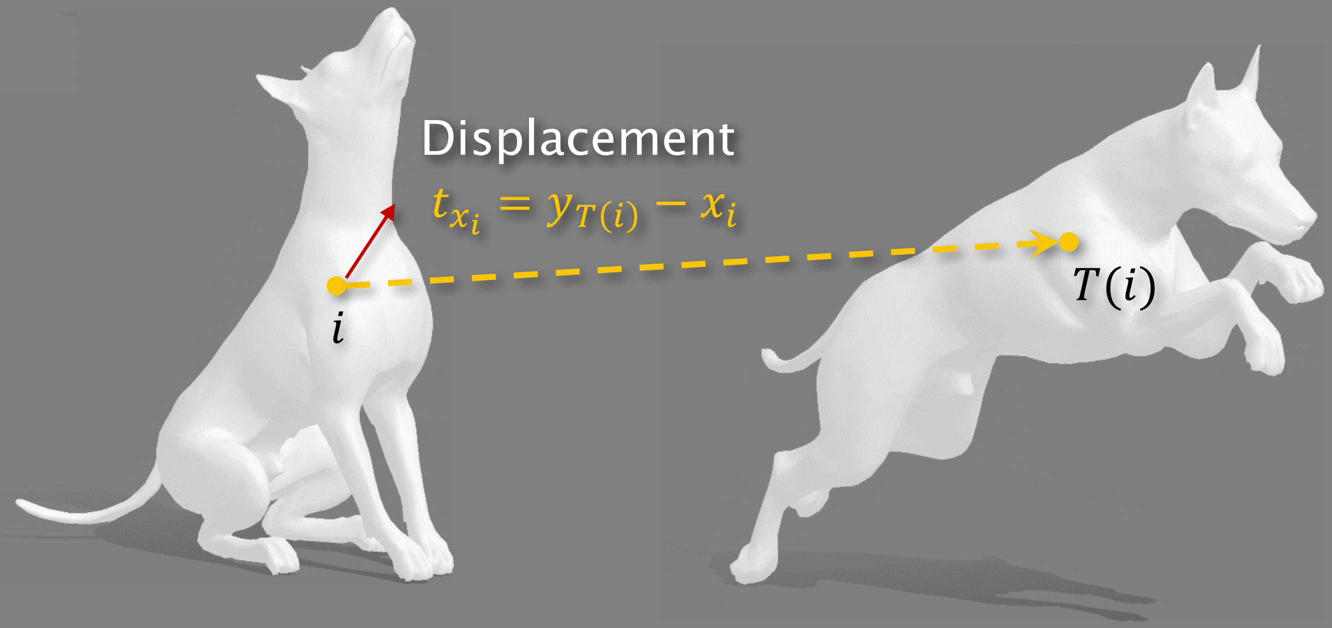
Φ_2

New energy

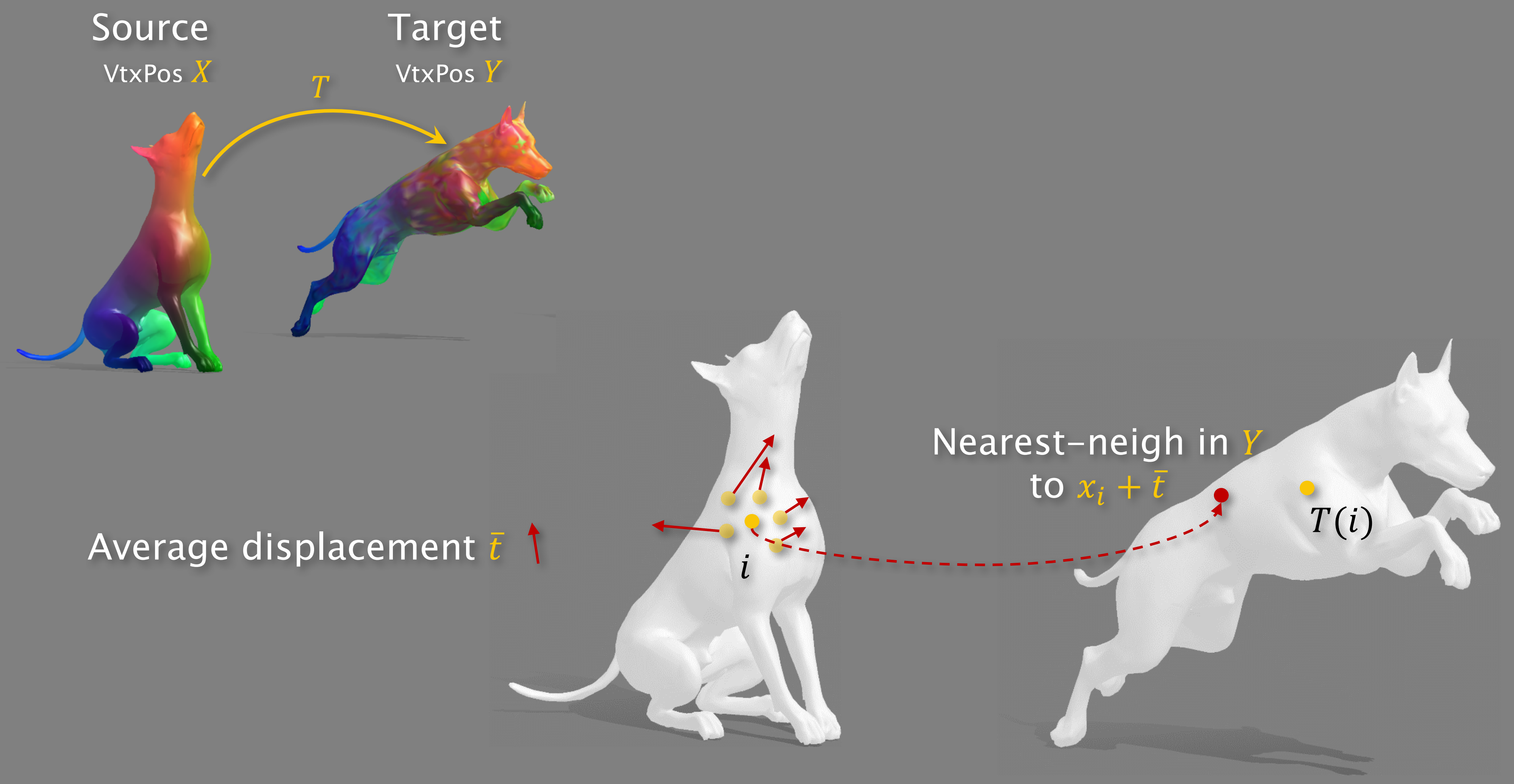
$$\begin{aligned} &\lambda_1\|\Phi_2 C_{12} - \Phi_1(T_{21},:)\|^2 \\ +\, &\lambda_2\|\Phi_2 C_{21} - \Phi_1(T_{12},:)\|^2 \\ +\, &\lambda_3\|\Phi_1 C_{11} - \Phi_1(T_{21} \circ T_{12},:)\|^2 \\ +\, &\lambda_4\|\Phi_2 C_{22} - \Phi_2(T_{12} \circ T_{21},:)\|^2 \end{aligned}$$

BCICP refinement – improve smoothness



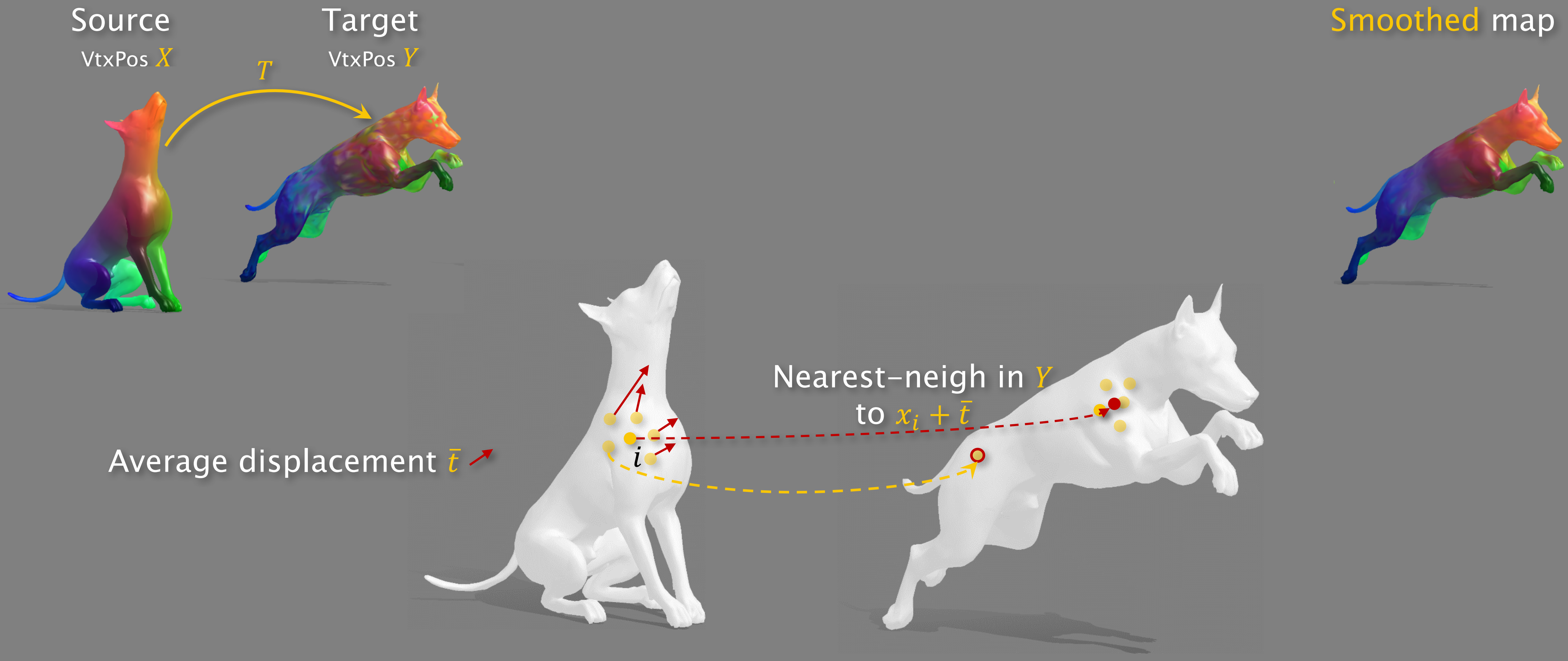


BCICP refinement – improve smoothness



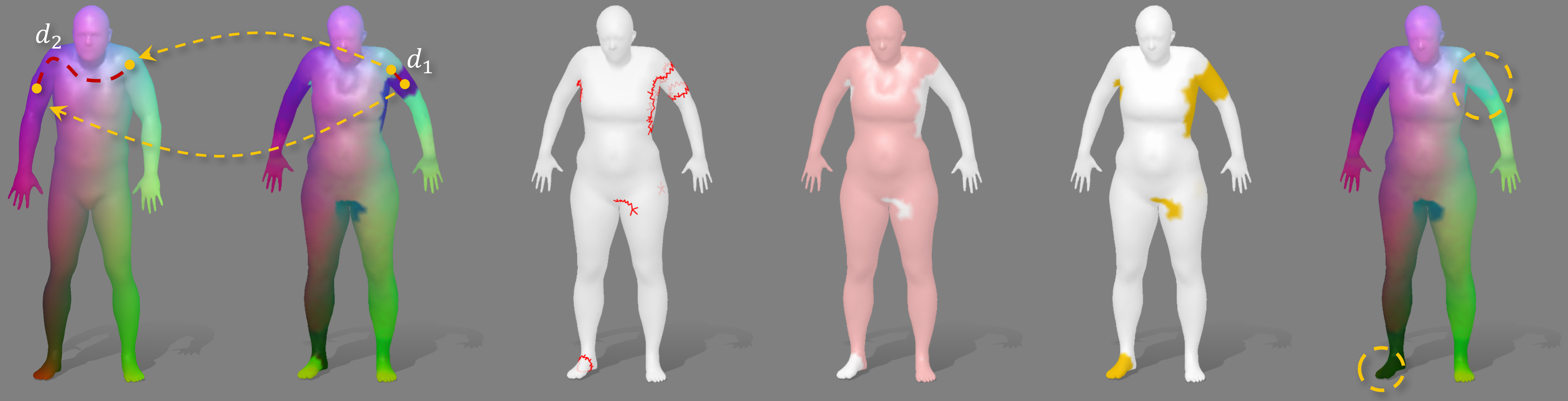
[Papazov and Burschka 2011]

BCICP refinement – improve smoothness



BCICP refinement – remove outliers

Source | Target | Edges with large distortion | Trustworthy vertices | Outlier vertices | Final map



Compute the edge distortion $r = d_2/d_1$

Remove these edges from the adjacency matrix of the mesh

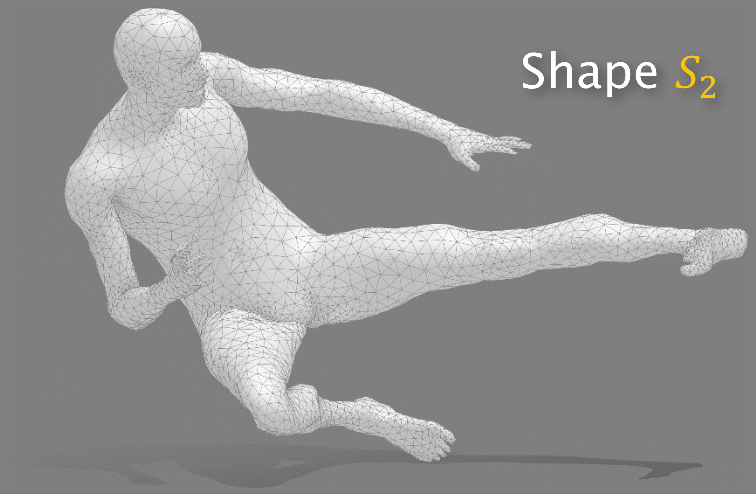
Find the largest connected component

Find the outlier region

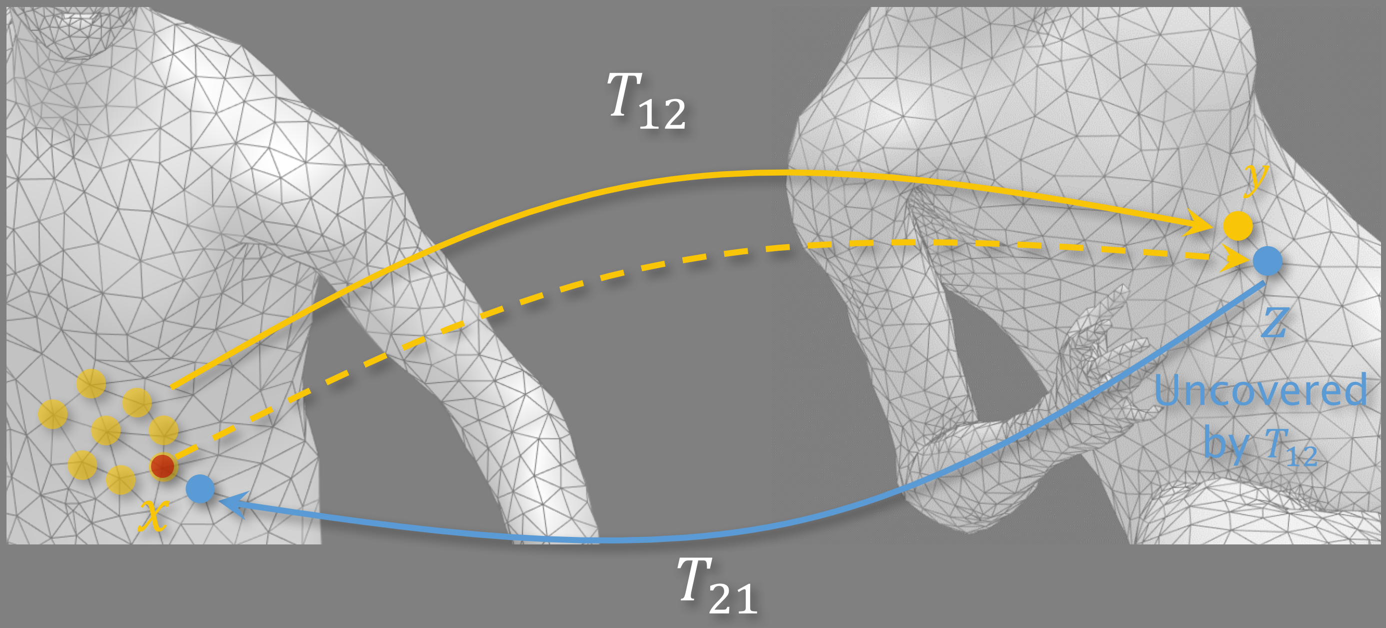
Outlier – find its NN in the "trustworthy" vertex set

BCICP refinement – improve coverage

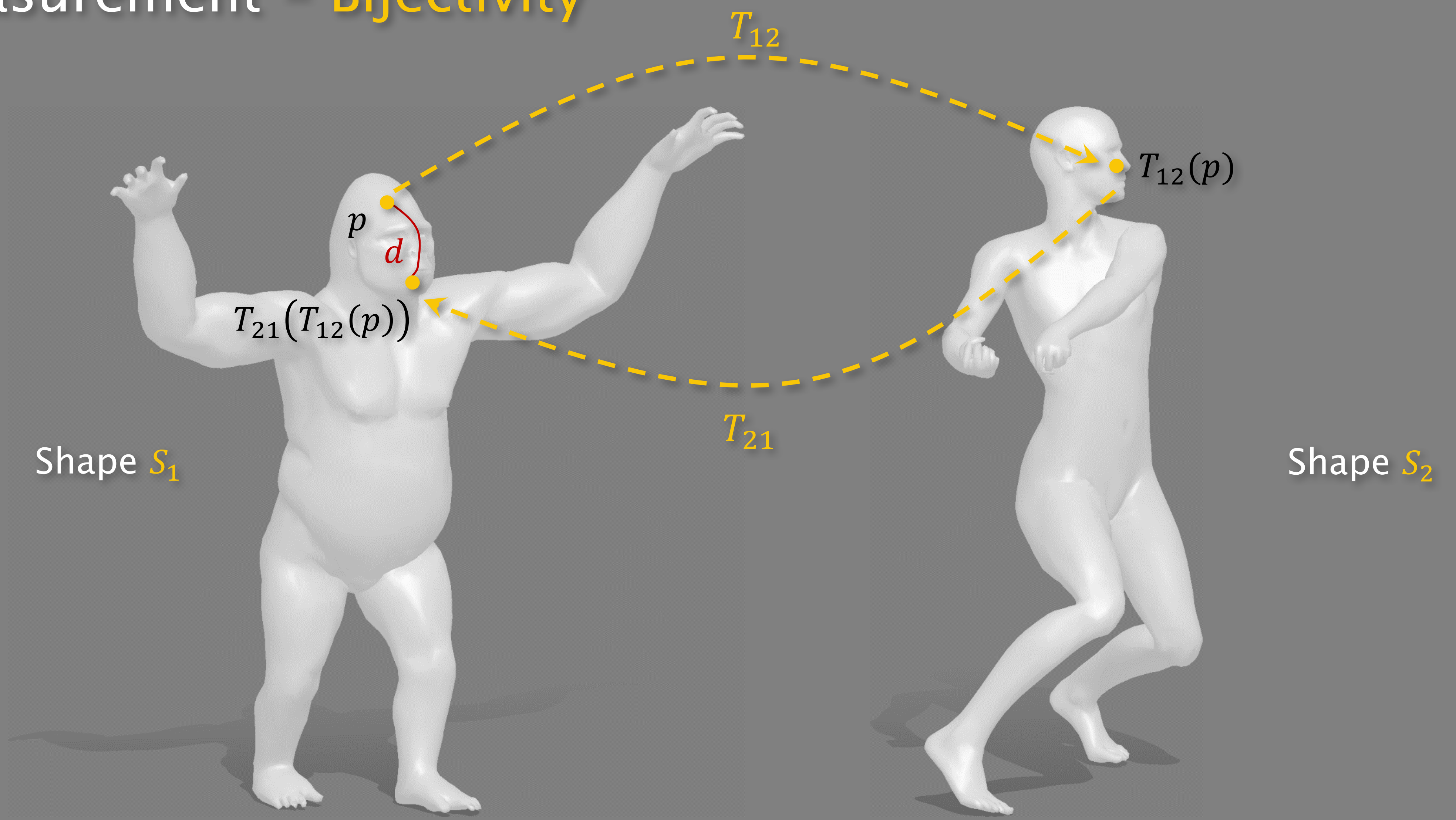
Shape S_1



Shape S_2

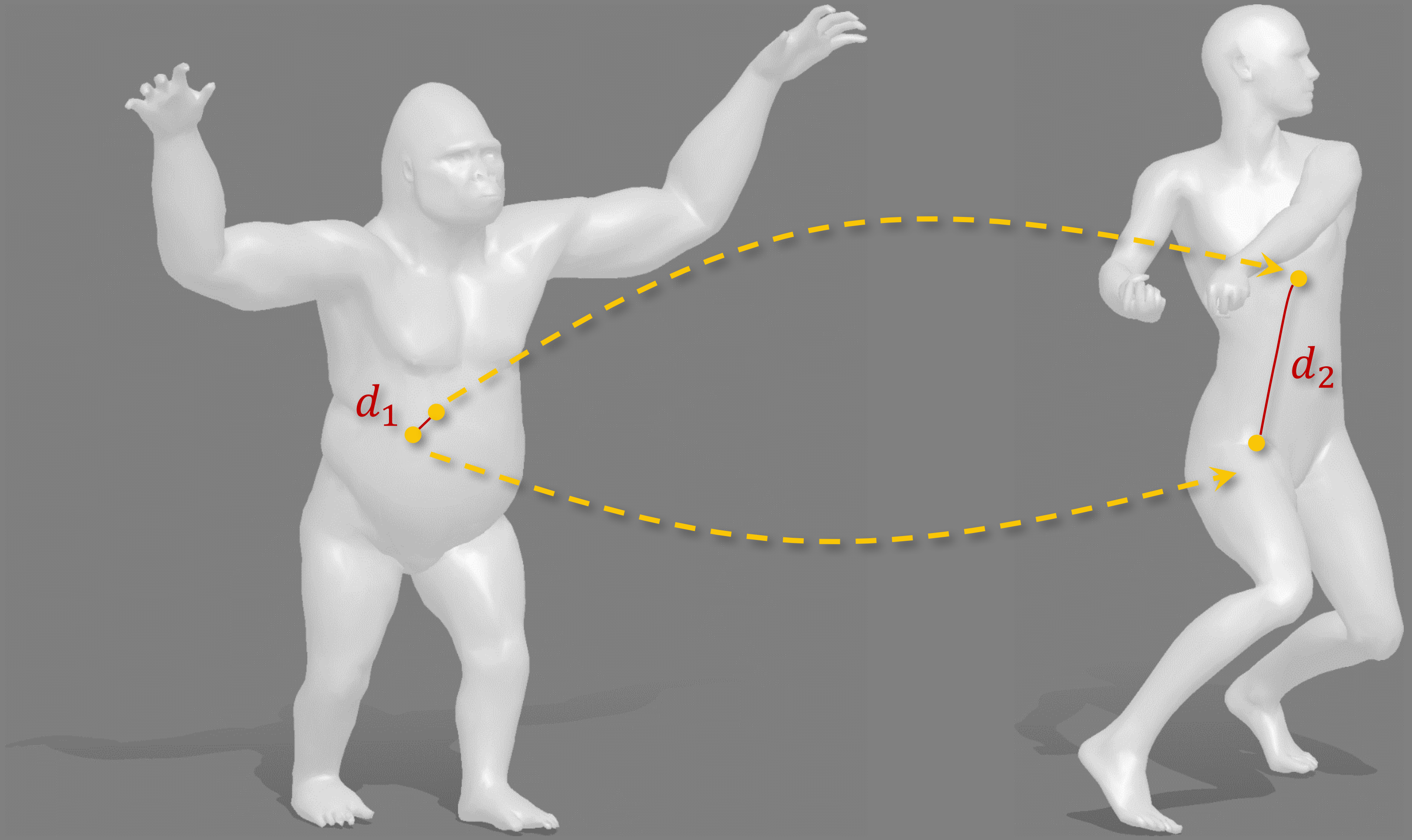


Measurement – Bijectivity



- d: geodesic distance between p and $T_{21}(T_{12}(p))$
- Measure the difference between $T_{21}(T_{12}(\cdot))$ and the identity map

Measurement – Smoothness



- $\frac{d_2}{d_1}$: edge distortion ratio to measure the smoothness

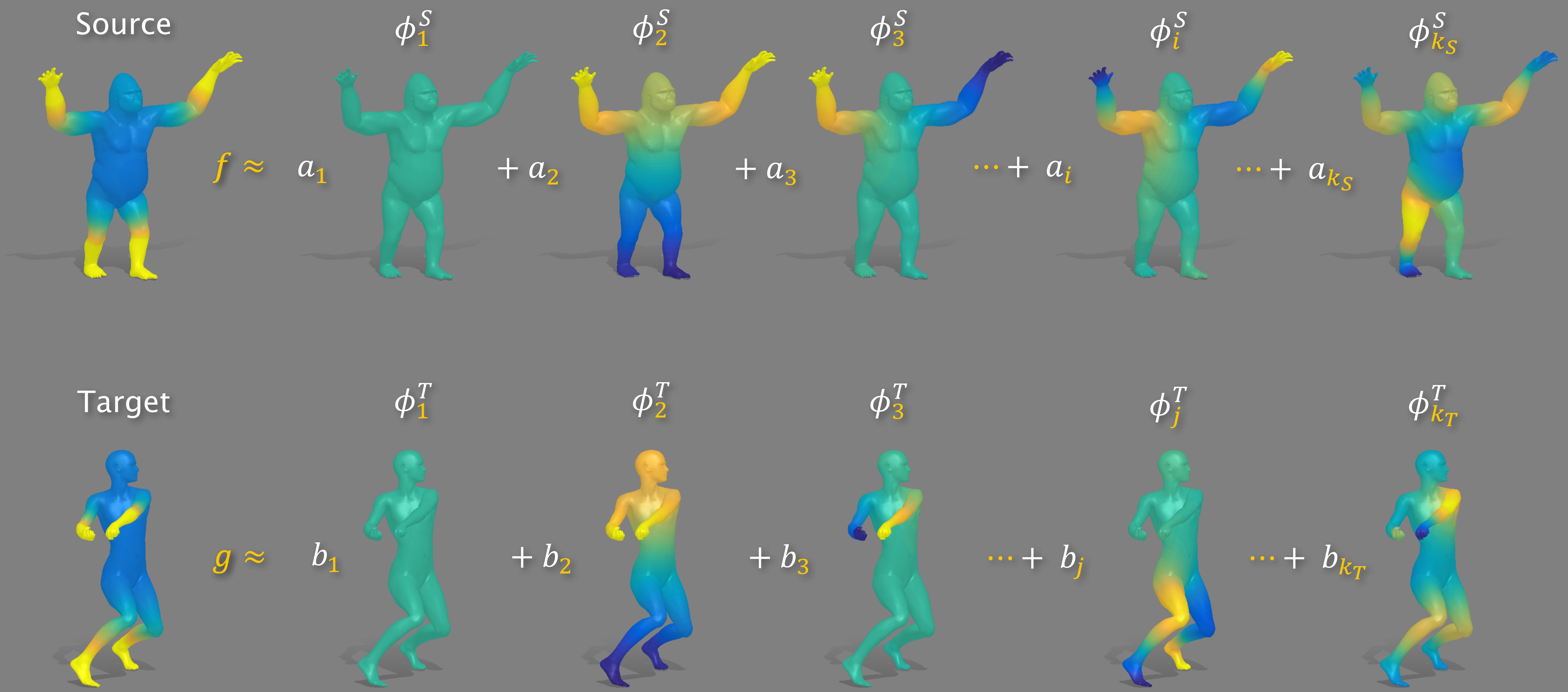
Refinement

Heuristic measurement
- High bijectivity
- High smoothness
  - No outliers
  - High coverage

Our solution – bijective and continuous ICP (BCICP)
- Refine the map simultaneously in the spectral and the spatial domain
- Improve the bijectivity, continuity, and coverage

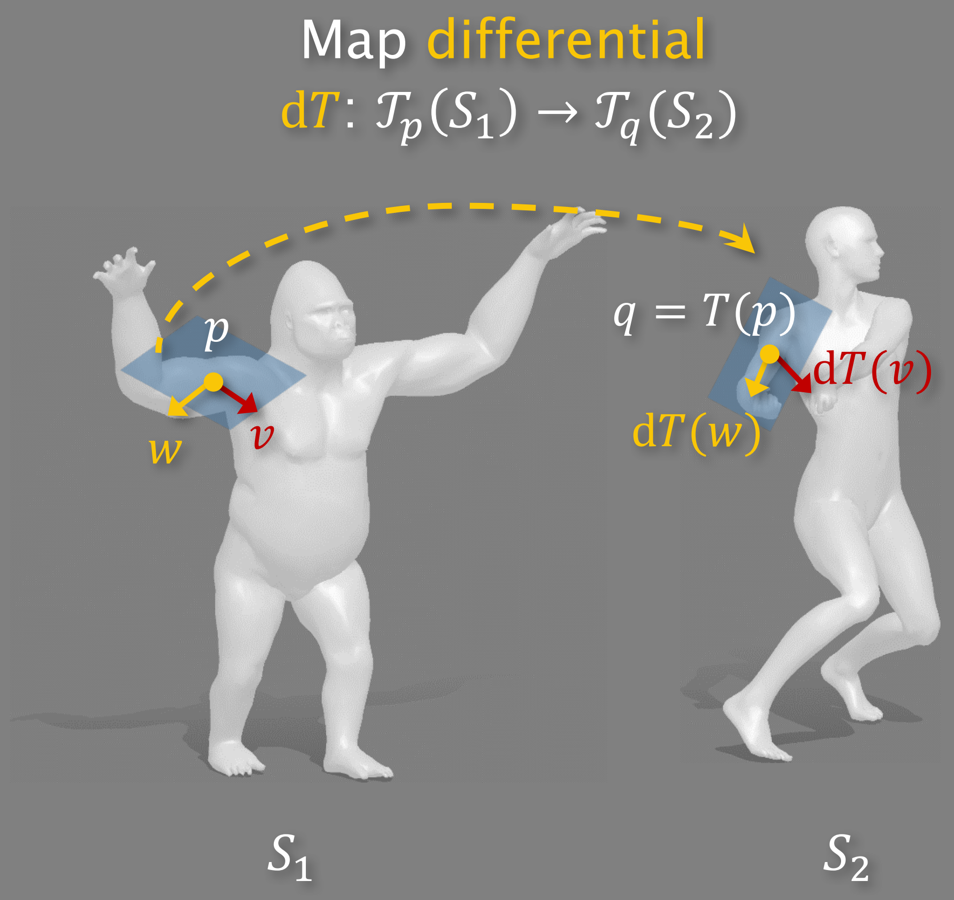
Q1: Functional map pipeline

Descriptors f/g

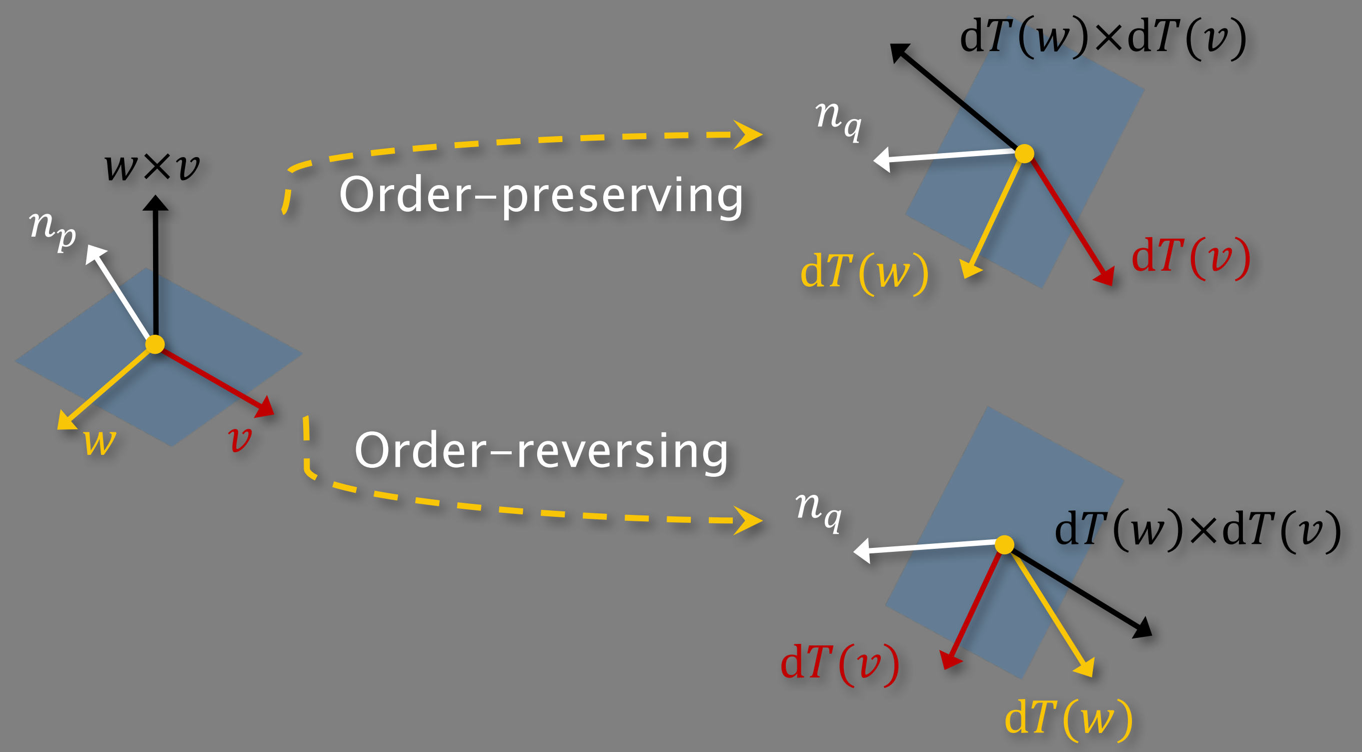


Q1: Functional map pipeline — Laplacian–Beltrami basis

Q2: Orientation-preserving operators



Frame at p: (w, v, n_p)
Frame at $q = T(p)$: $\left(dT(w), dT(v), n_{T(p)}\right)$



$(w \times v)^T n_p$ should have the same sign as $\left(dT(w) \times dT(v)\right)^T n_{T(p)}$

Q1: Functional map pipeline

- First introduced by Ovsjanikov et al in 2012 : "*Functional Maps: A Flexible Representation of Maps between Shapes*"
- Map a function defined on the source shape to another function on the target
- The functions defined on the source/target shape are represented in a compressed form using the Laplacian-Beltrami basis