Joint Graph Layouts

for Visualizing Collections of Segmented Meshes

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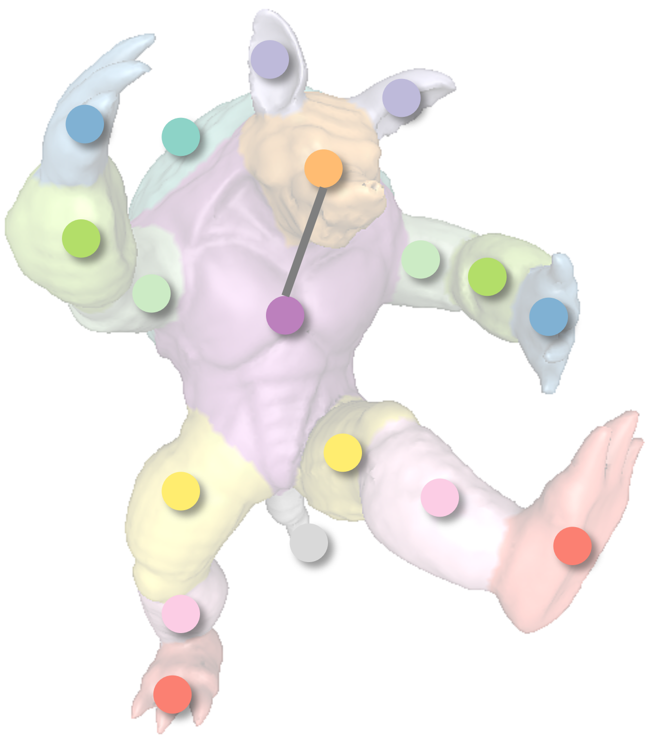
Graph Representation

Applications

- Social networks
- Protein–protein interaction
- Organizational hierarchy
- ……
- Connectivity graph of segmented 3D shapes

Graph Representation

Segmented 3D shape



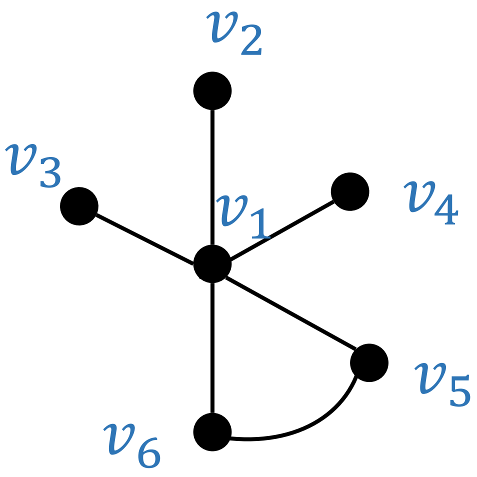
Node: each segment
Edge: if connected

Graph representation

Graph Drawing

Problem formulation

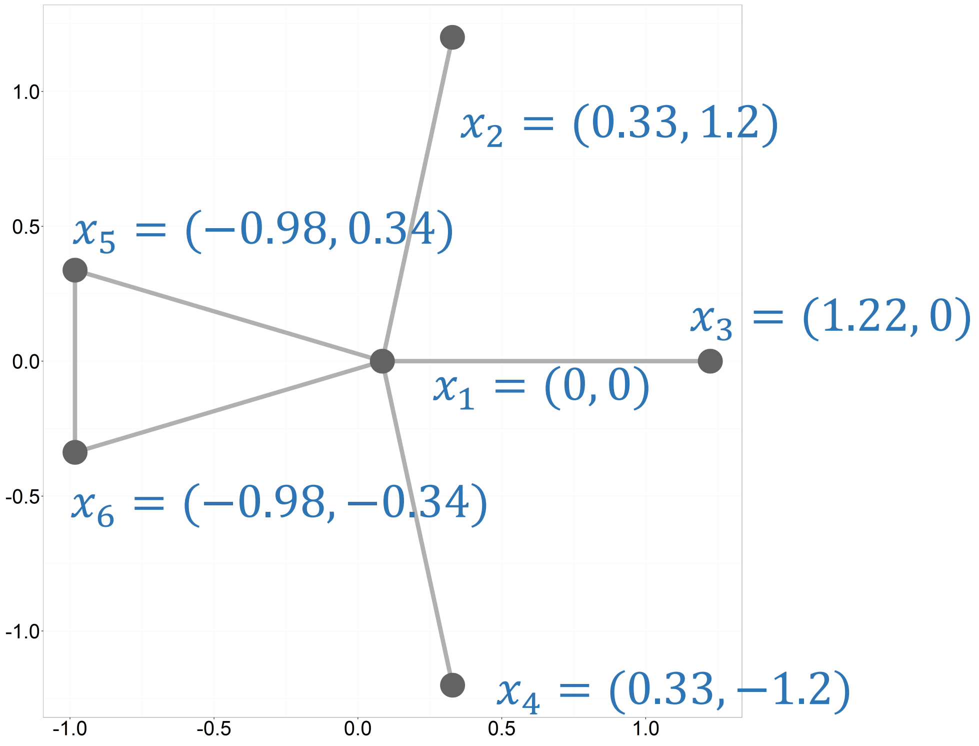
Graph $G = (V, E)$



Vertex $V = \{v_1, \cdots, v_6\}$

Edge $E = \{e_1, \cdots, e_6\}, e_k = (v_i, v_j)$

Embedding X



$x_i \in R^2$ the position for vertex v_i

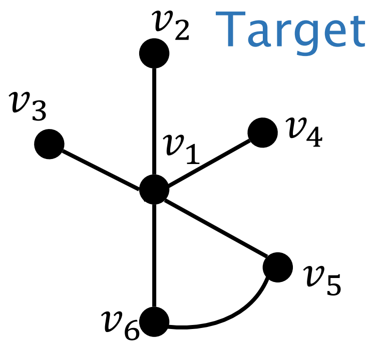
Graph Drawing

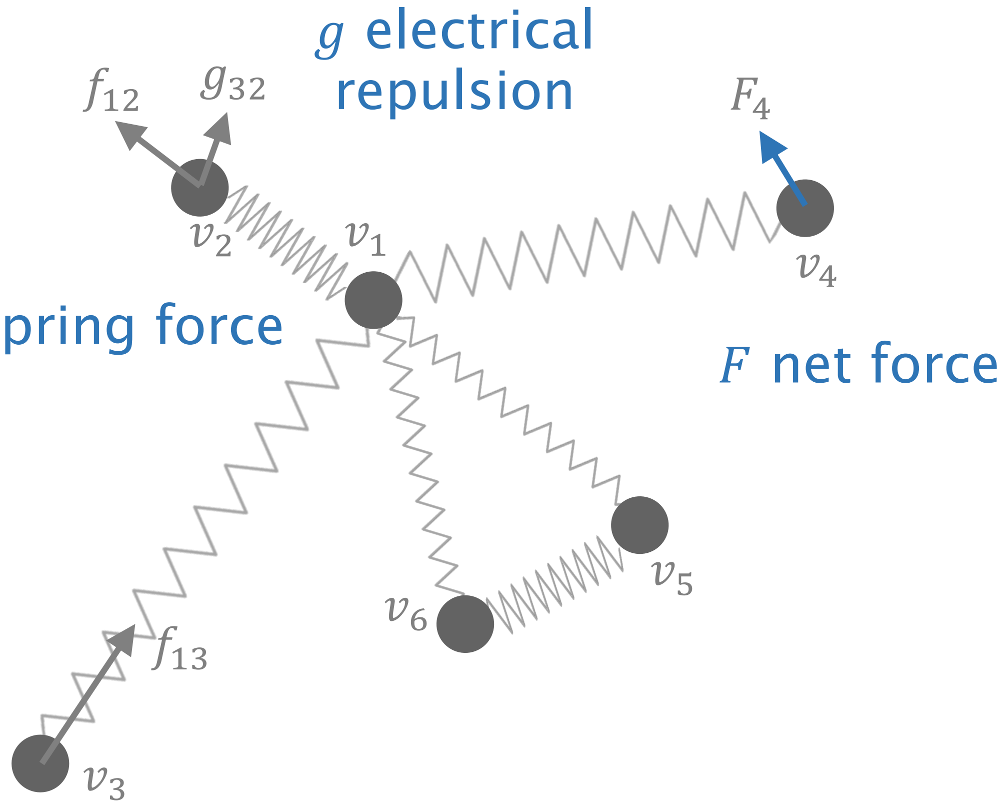
State-of-the-art

- Force-directed graph drawing
  - graph → force system, equilibrium configurations → embedding
- Spectral drawing
  - nodes that are connected to each other should have closer position
- Multidimensional Scaling (MDS)
  - Preserve pair-wise graph distances.
- ……

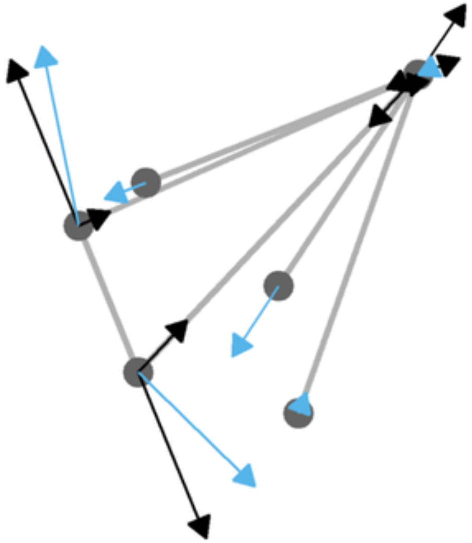
Graph Drawing

Force-directed method





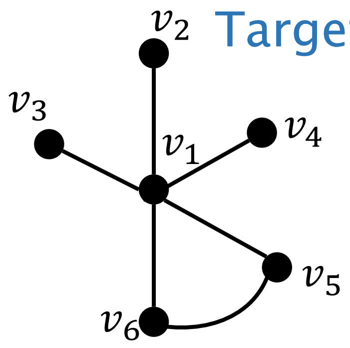
Force system with springs and charged particles



Find equilibrium

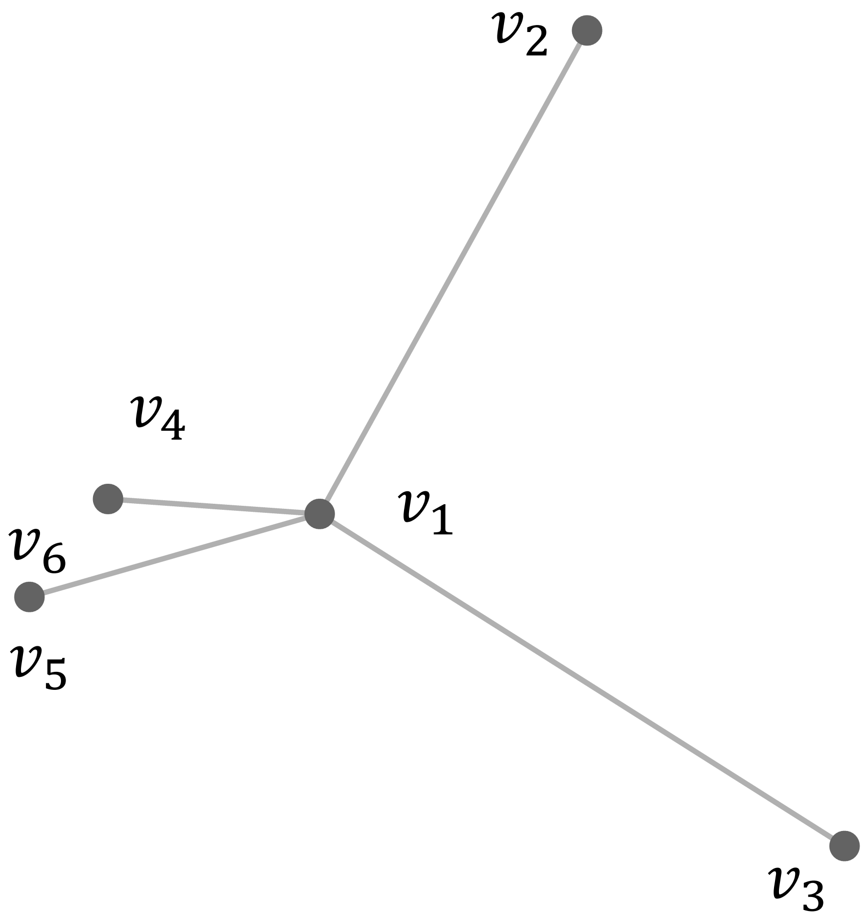
Graph Drawing

Spectral drawing



Objective: connected notes should be close–by

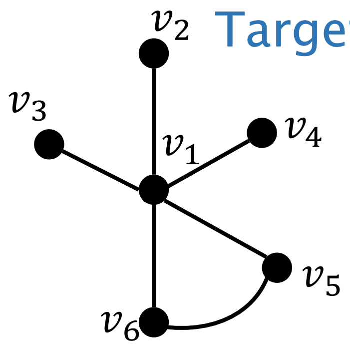
$$E = \sum_{v_i \sim v_j} \left\| x_i - x_j \right\|^2$$



- $X^T X = I$: avoid trivial solution
- $E = \mathrm{trace}(X^T L X)$, where L is the graph Laplacian
- Close–form solution: Eigenvectors of L

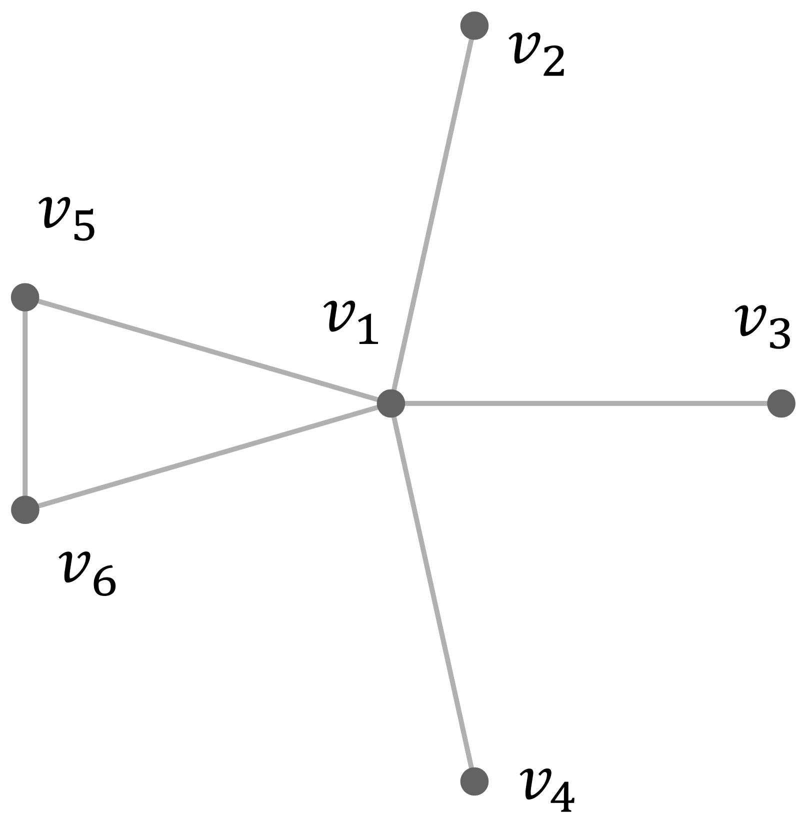
Graph Drawing

Multidimensional Scaling (MDS)



Objective: preserve pair-wise graph distances

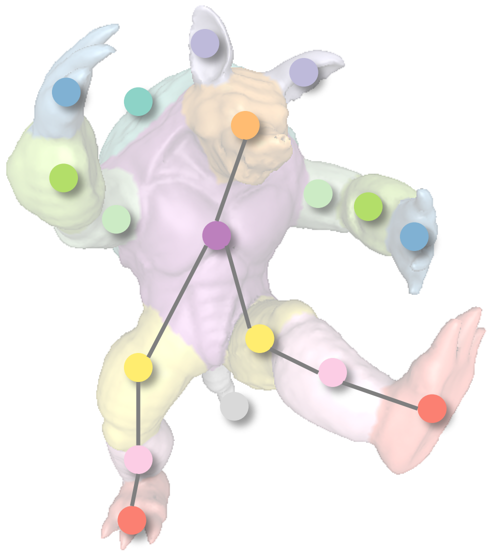
$$E = \sum_{i=1}^{n} \sum_{j=1}^{n} w_{ij} \left(\| x_i - x_j \| - d_{ij} \right)^2$$



- Embedded Euclidean distance $\|x_i - x_j\|$ close to graph distance d_{ij}
- Non-convex problem – Stress Majorization method

Graph Drawing

Single graph

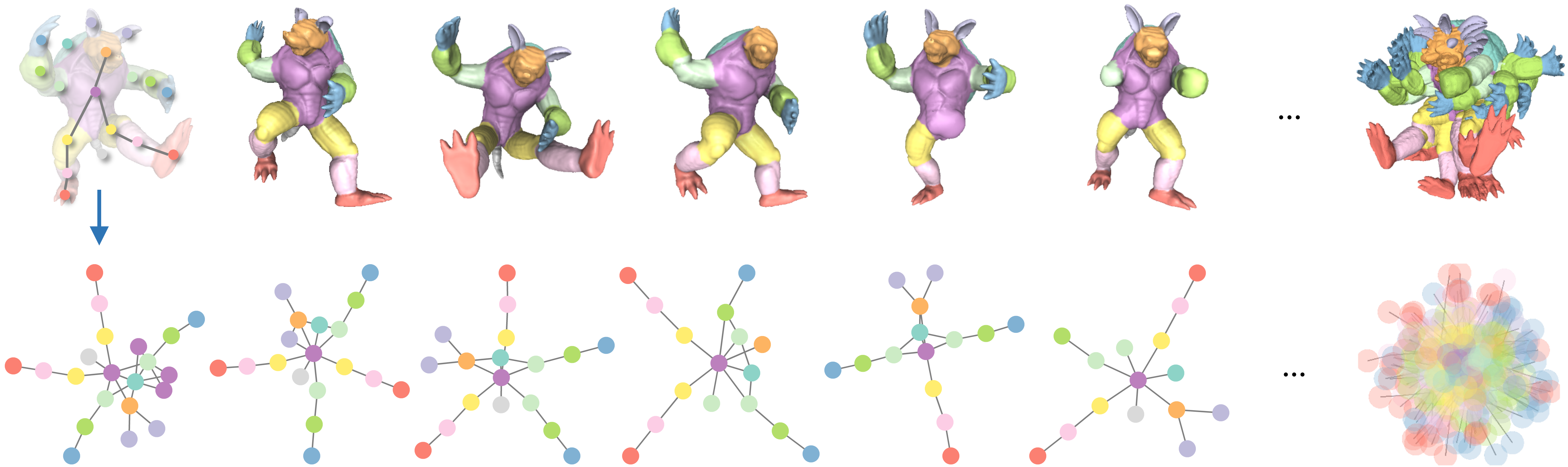


Connectivity

2D embedding

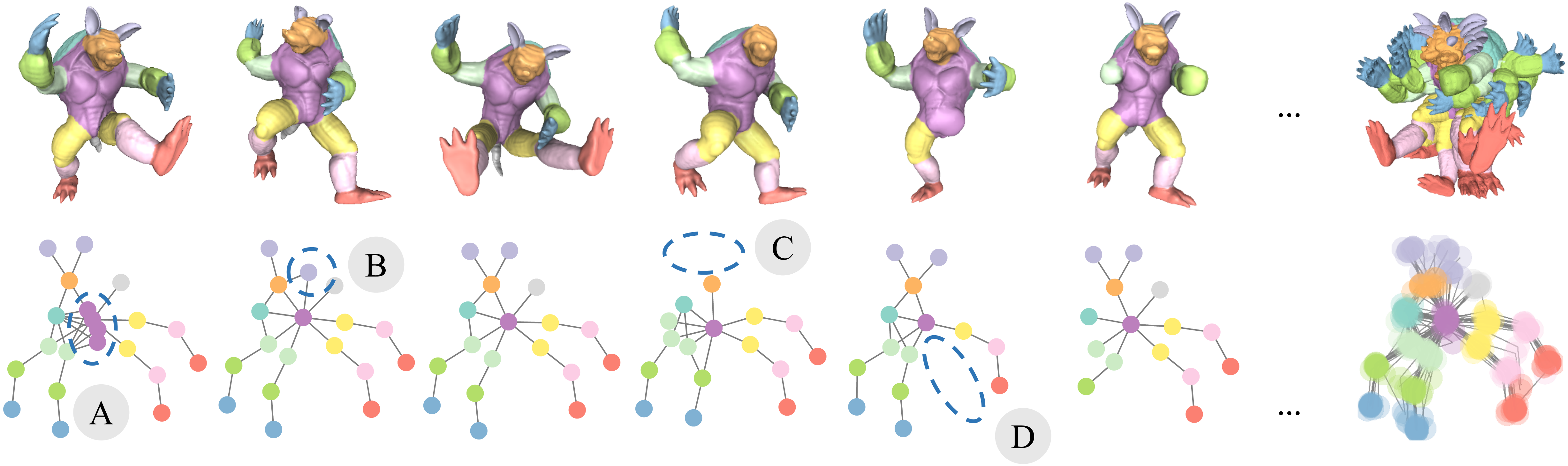
Graph Drawing

Multiple graphs



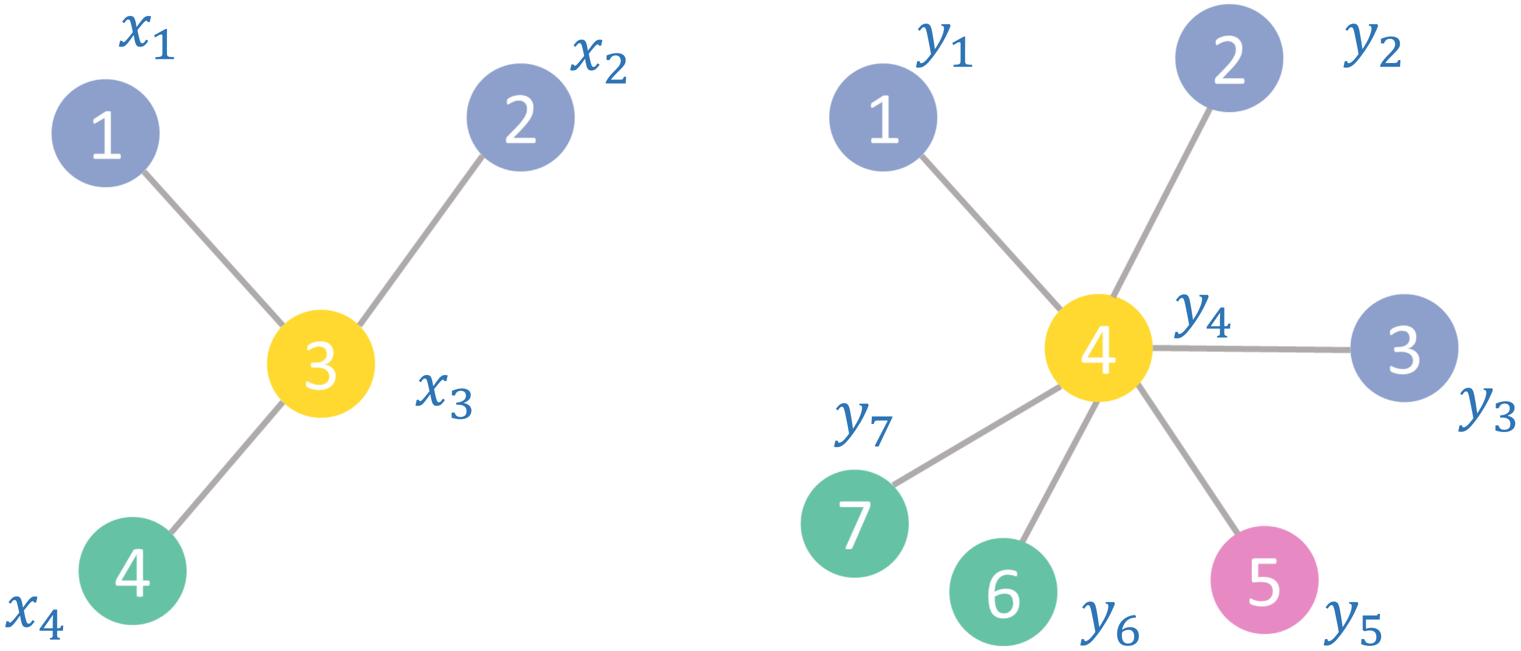
Joint Graph Layouts

- For each graph, the graph structure is preserved
- + Consistency: nodes from different graphs with the same label are in a nearby location



Joint Graph Layouts

Correspondences



G_p

G_q

Embedding: $X^{(p)} = (x_1, \cdots, x_4)$

Embedding: $X^{(q)} = (y_1, \cdots, y_7)$

Joint Graph Layouts

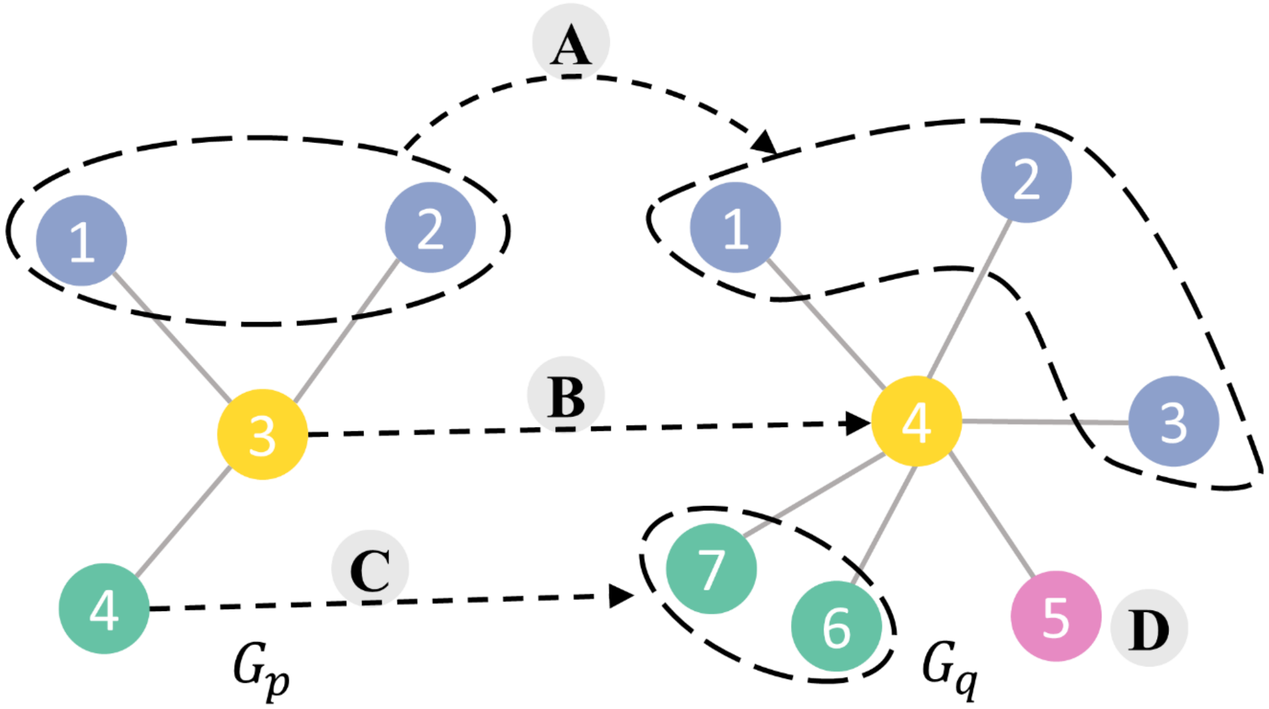
Correspondences

(A): $\frac{1}{2}(x_1 + x_2) \approx \frac{1}{3}(y_1 + y_2 + y_3)$

(B): $x_3 \approx y_4$

(C): $x_4 \approx \frac{1}{2}(y_6 + y_7)$

$$S_{pq} X^{(p)} \approx T_{pq} X^{(q)}$$



where

$$S_{pq} = \begin{pmatrix} 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad T_{pq} = \begin{pmatrix} 1/3 & 1/3 & 1/3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/2 & 1/2 \end{pmatrix}$$

Joint Graph Layouts

Formulation

Given a set of $\{G_k\}_{k=1}^{n}$, find the embedding $\{X^{(k)}\}$ such that

- For each graph G_k, graph structure is preserved
- For each pair of graphs (G_p, G_q), the correspondences are preserved $S_{pq}X^{(p)} \approx T_{pq}X^{(q)}$.

Joint Graph Layouts

Formulation

$$\min_{X^{(k)}} \lambda_1 E_1 + \lambda_2 E_2 + \lambda_3 E_3$$

- Smoothness term

$$E_1(X) = \sum_k \sum_{v_i \sim v_j} \left\| X_i^{(k)} - X_j^{(k)} \right\|_F^2$$

- Consistency term

$$E_2(X) = \sum_{1 \le p < q \le n} \left\| \mu_{pq} \left(S_{pq} X^{(p)} - T_{pq} X^{(q)} \right) \right\|_F^2$$

- Distance preservation term
  - $E_3(X) = \sum_{k=1}^{n} \sum_{1 \le i < j \le m_k} \lambda_{ij}^{(k)} \left(\left\| X_i^{(k)} - X_j^{(k)} \right\| - \delta_{ij}^{(k)} \right)^2$

Joint Graph Layouts

Algorithms

- Objectives

$$\min_{X^{(k)}} \lambda_1 E_1 + \lambda_2 E_2 + \lambda_3 E_3$$

- Algorithms
  - Step 01: spectral initialization

$$X_{\mathrm{ini}} = \operatorname*{argmin}_{X^T X = I} \lambda_1 E_1 + \lambda_2 E_2$$

  - Step 02: stress majorization (starts with X_{ini})

$$X^* = \operatorname{argmin}\ \lambda_1 E_1 + \lambda_2 E_2 + \lambda_3 E_3$$

Joint Graph Layouts

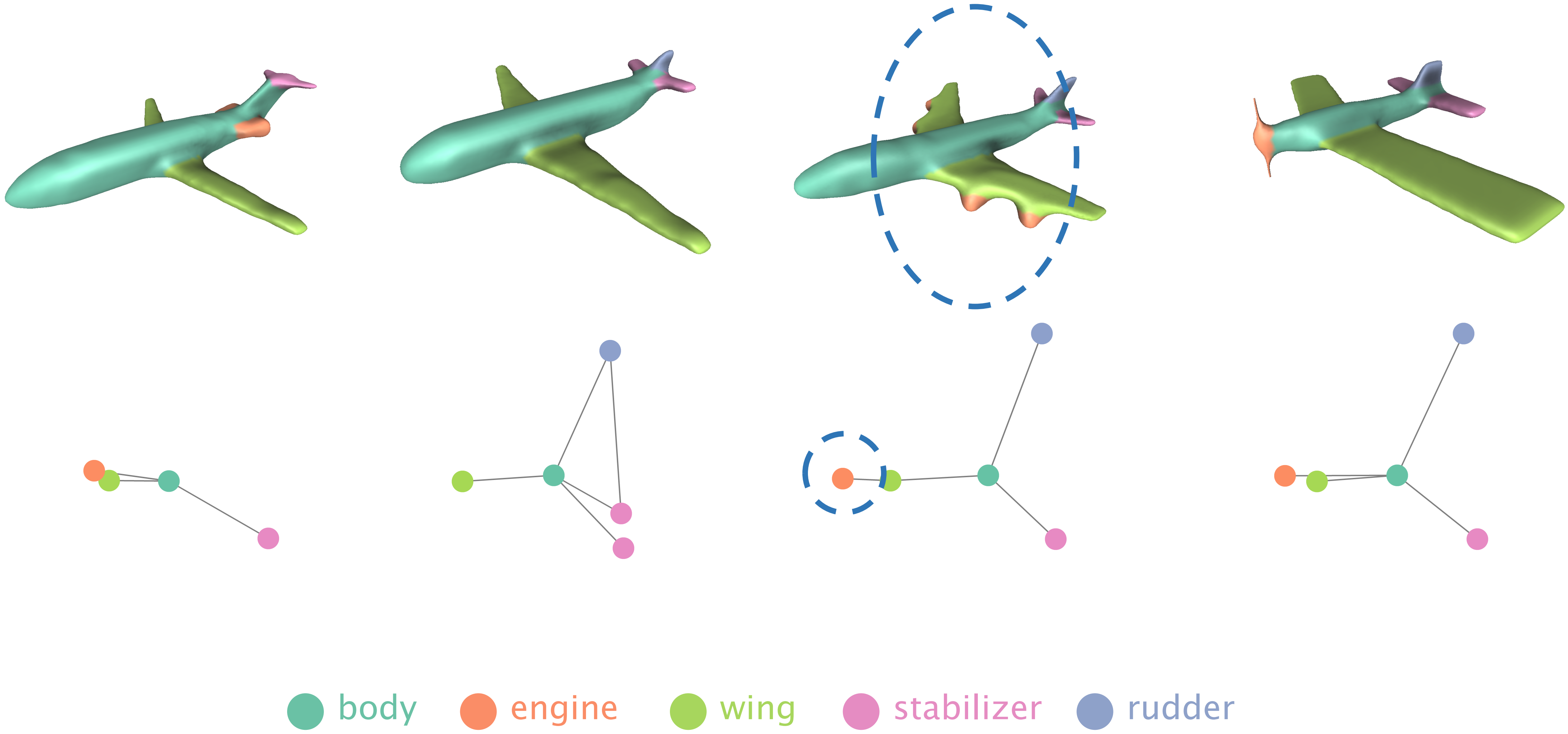
Spectral Initialization

$$X_{\text{ini}} = \underset{X^T X = I}{\operatorname{argmin}} \lambda_1 E_1 + \lambda_2 E_2 = \underset{X^T X = I}{\operatorname{argmin}} \operatorname{trace}(\, X^T W X \,)$$

X_{ini} has close-form global minimizer: the eigenvectors corresponding to the first two smallest eigenvalues of W.

Joint Graph Layouts

Spectral Initialization



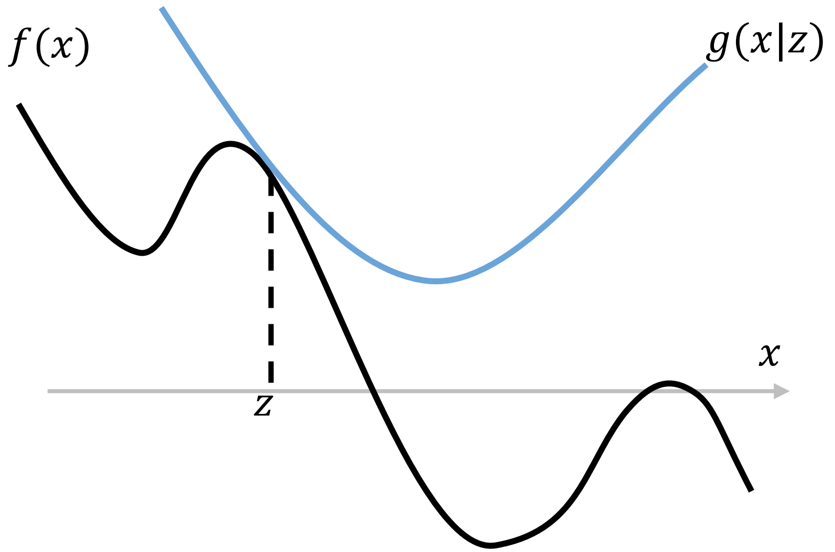
Joint Graph Layouts

Stress majorization

Definition. $g(x|z)$ is a majorizing function for $f(x)$ if:

1) $g(x|z) \geq f(x), \forall x$

2) $g(z|z) = f(z)$



Joint Graph Layouts

Stress majorization

Algorithm.

Input: $f(x), g(x|z), x_{\text{ini}}$

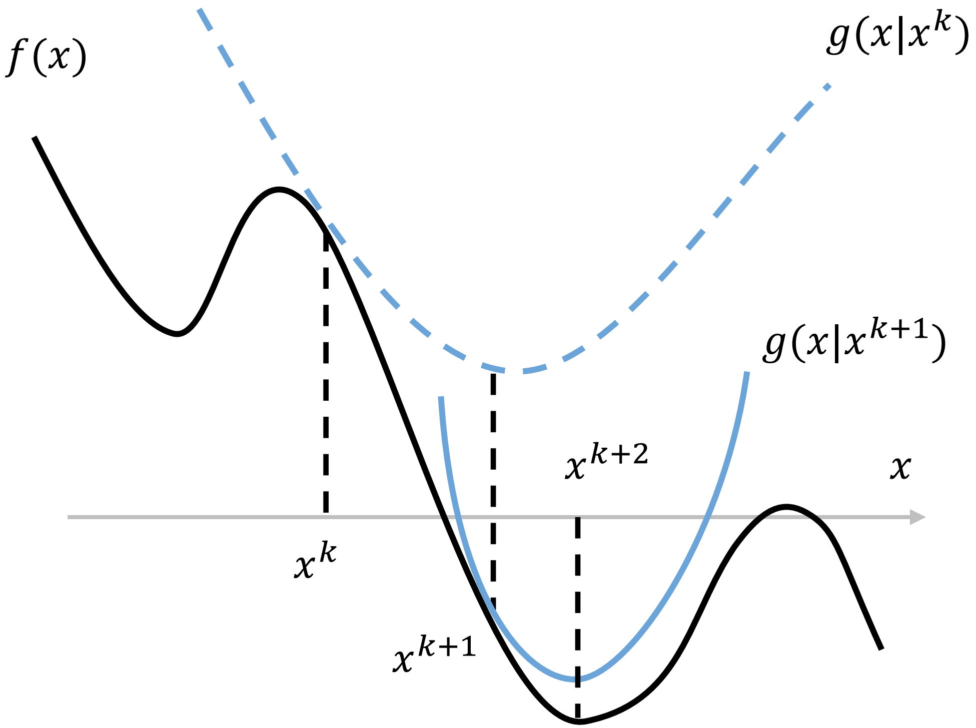
Output: x^* -- a local minimum of $f(x)$

For $k = 1,2,\ldots$

Solve $x^k = \text{argmin } g(x|x^{k-1})$

If $\|x^k - x^{k-1}\| \leq \epsilon$, return $x^* = x^k$

end



Joint Graph Layouts

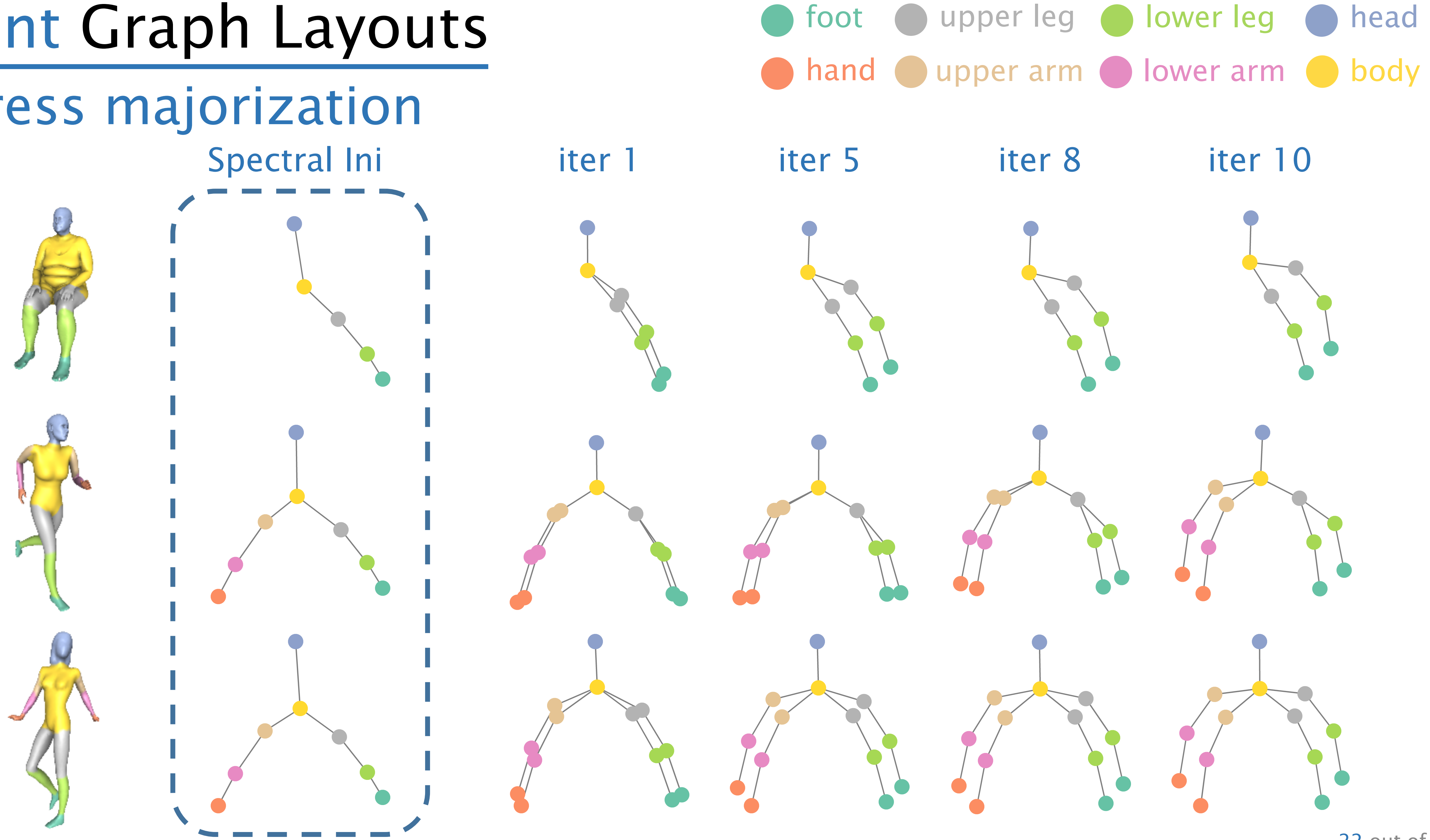
Stress majorization

Proposition. There exists a majorizing function $g(X|Z)$ for the total energy

$$F(X) = \lambda_1 E_1(X) + \lambda_2 E_2(X) + \lambda_2 E_3(X)$$

Joint Graph Layouts

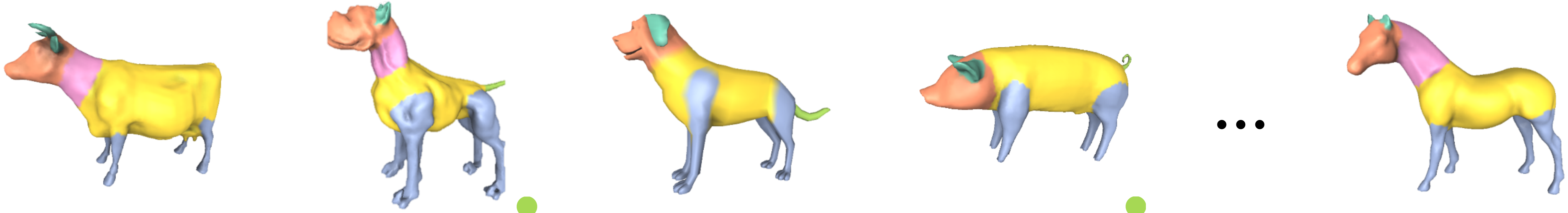
Stress majorization



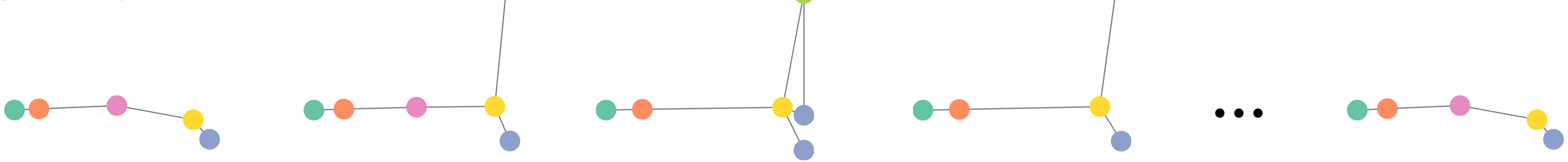
Joint Graph Layouts

Algorithms

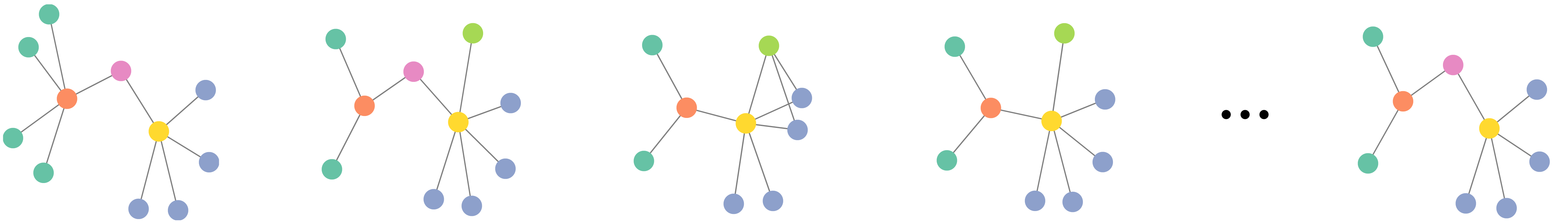
ear/horn · head · tail · neck · leg · torso



Step 01: spectral initialization

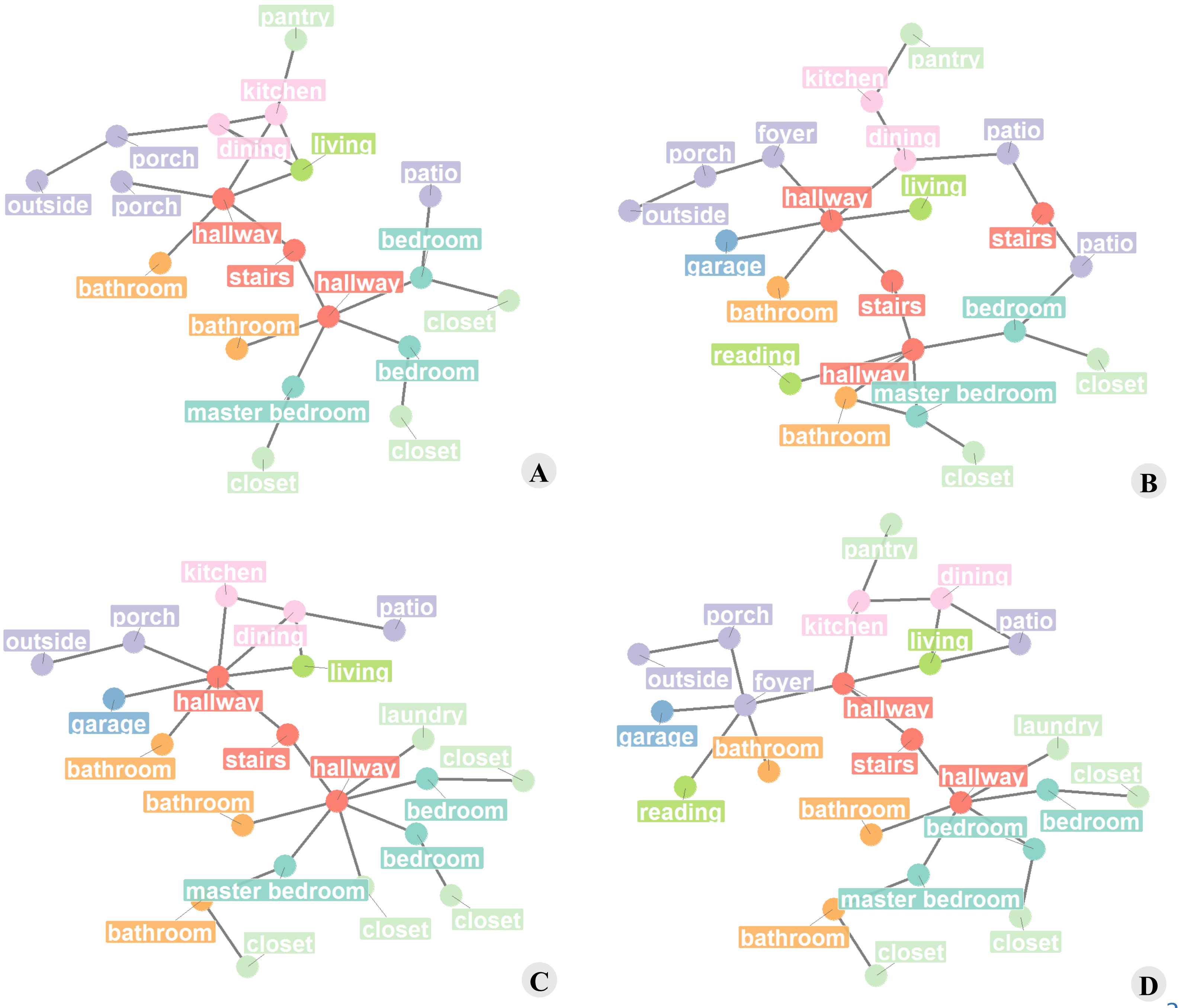


Step 02: stress majorization

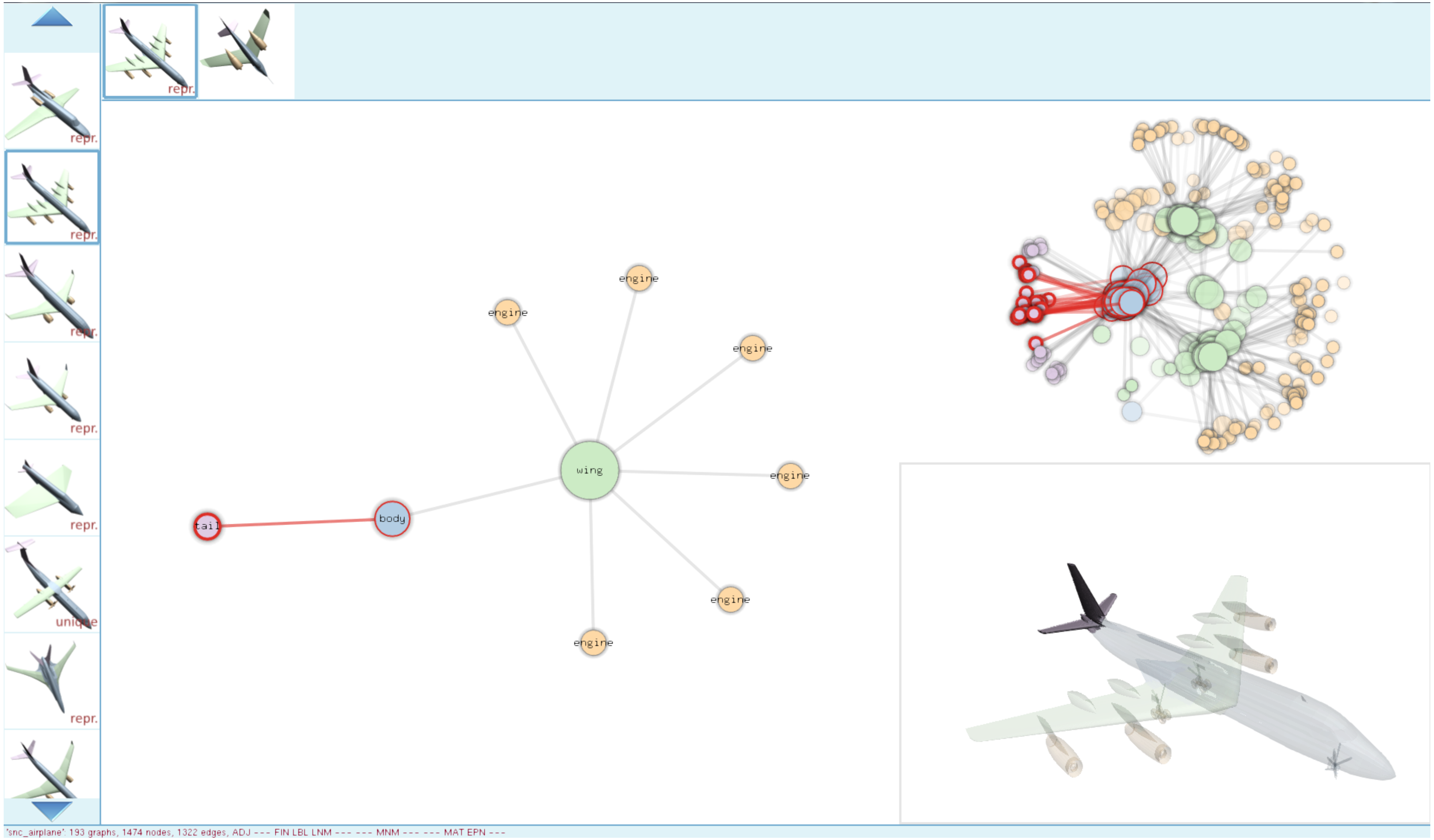


Results

Floor plans

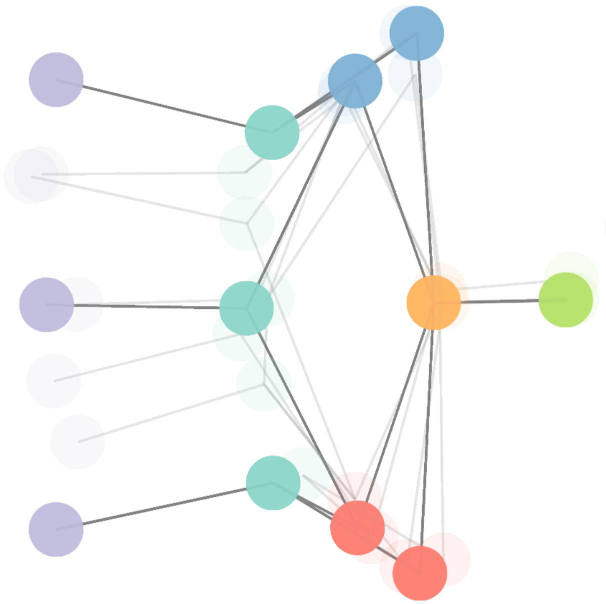


User Interface



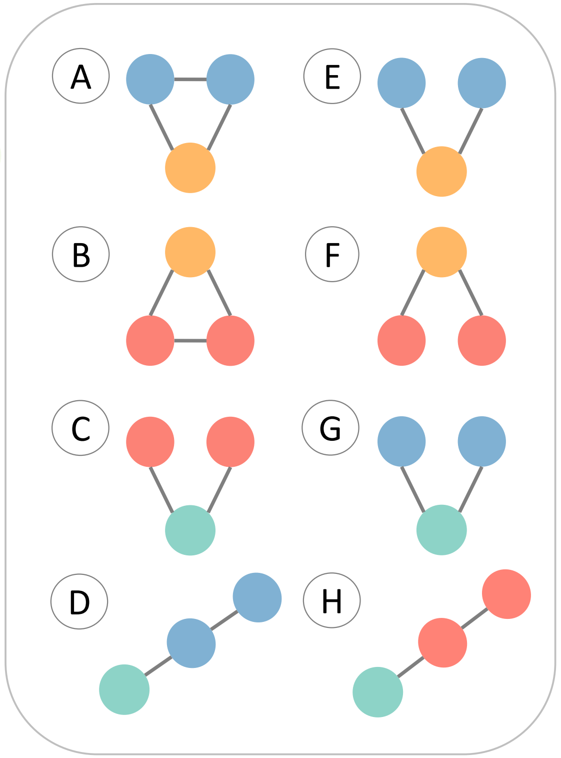
User study

Q1: Are the graphs in the collection the same or not?
Q2: Which graph is different from the rest?
Q3: Which graph collection has a larger variability?
……
Q7: Which subgraphs appear in the dominant structure of the given collection?

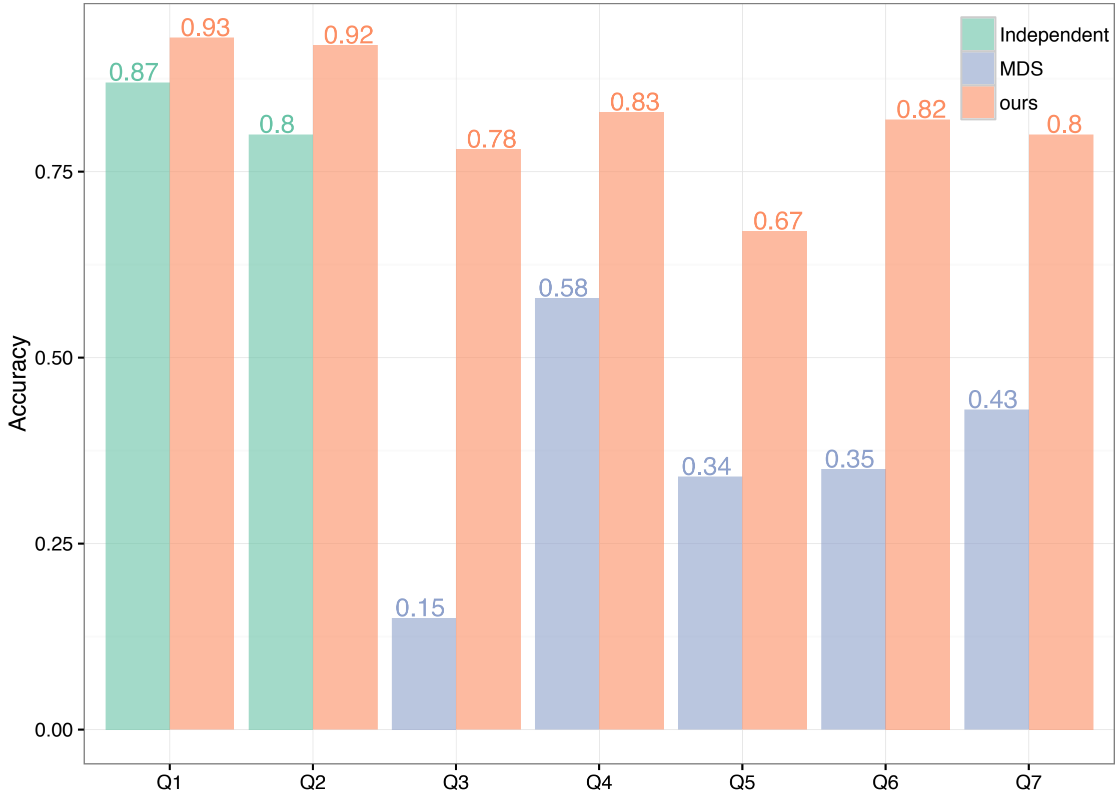


MDS

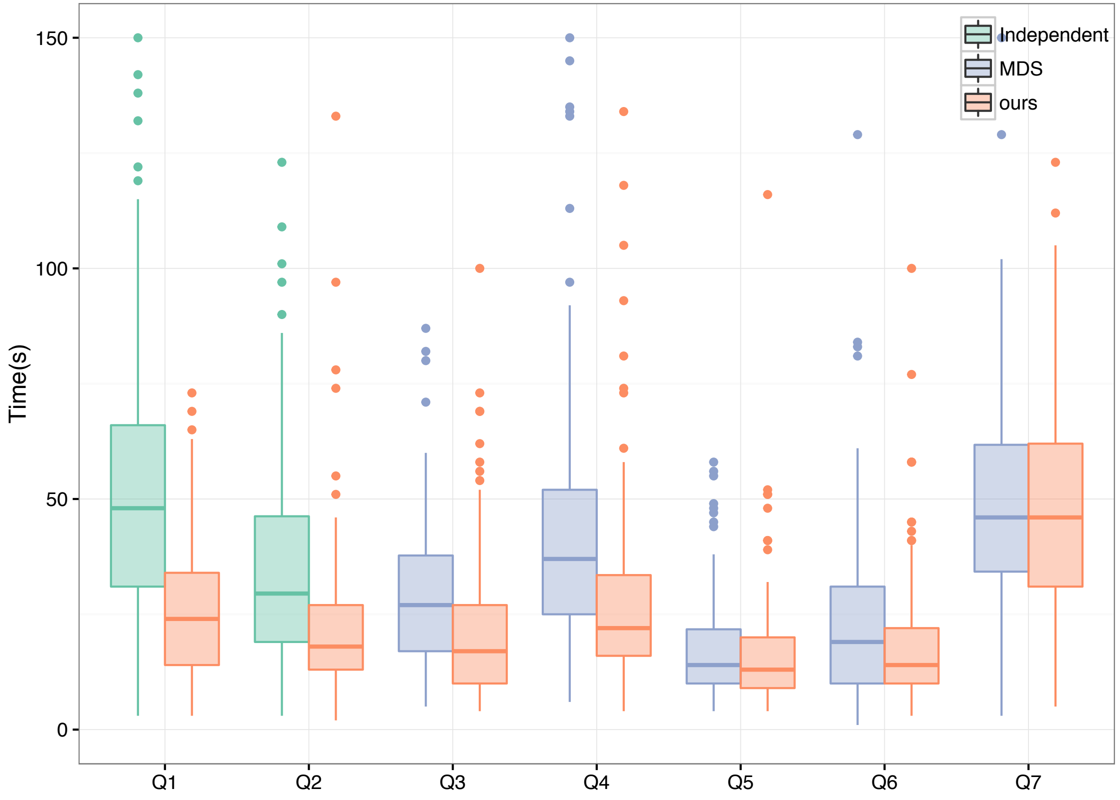
Ours



User study



Accuracy



Time

Summary

- Objective
  - Consistently embed a set of graphs
- Formulation
  - Smoothness term
  - Consistency term
  - Distance-preservation term
- Algorithms
  - Spectral initialization: Eigen-decomposition
  - Stress-majorization: solving a linear system for each iteration

Thanks for your attention

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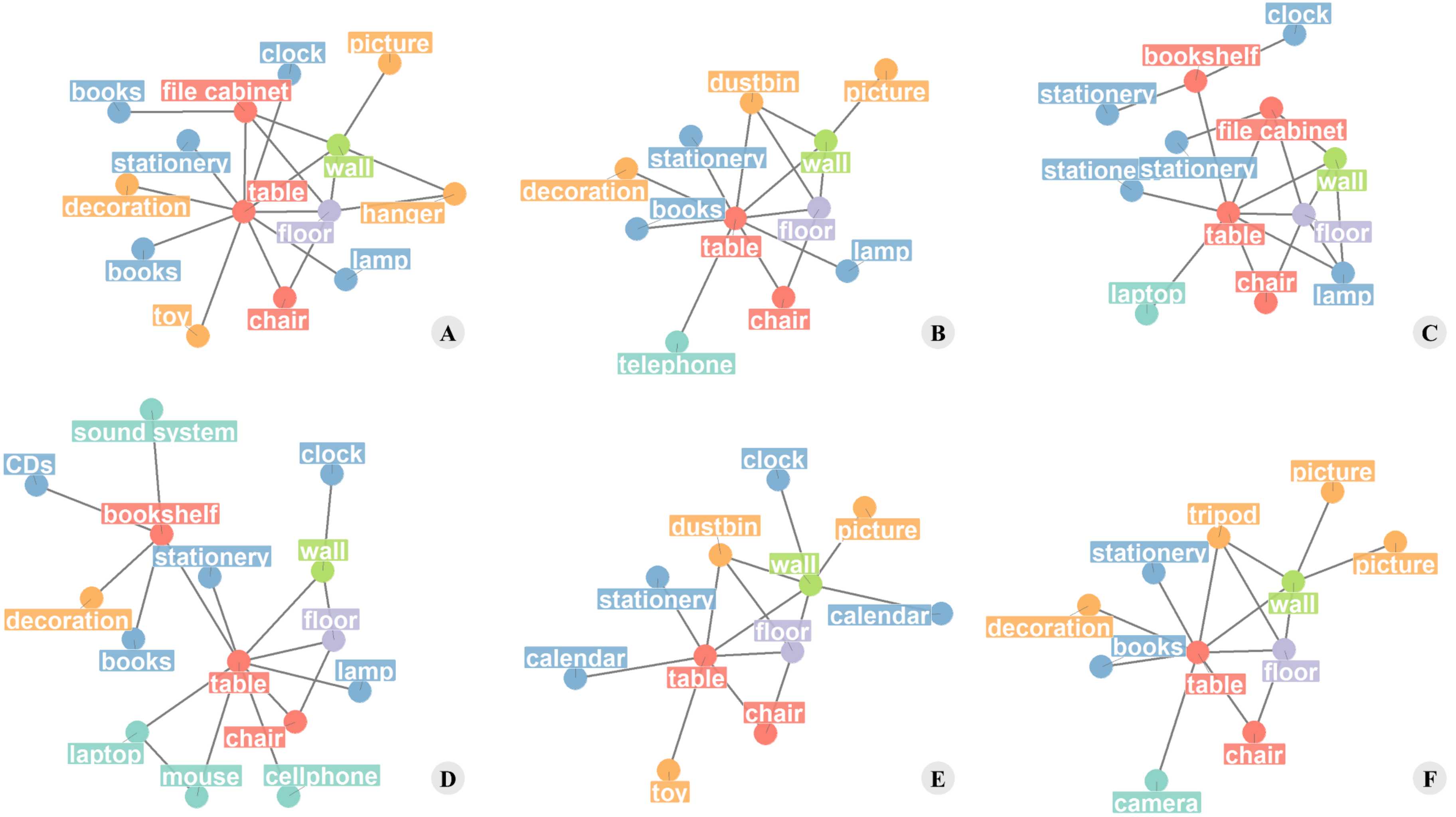
Sample code

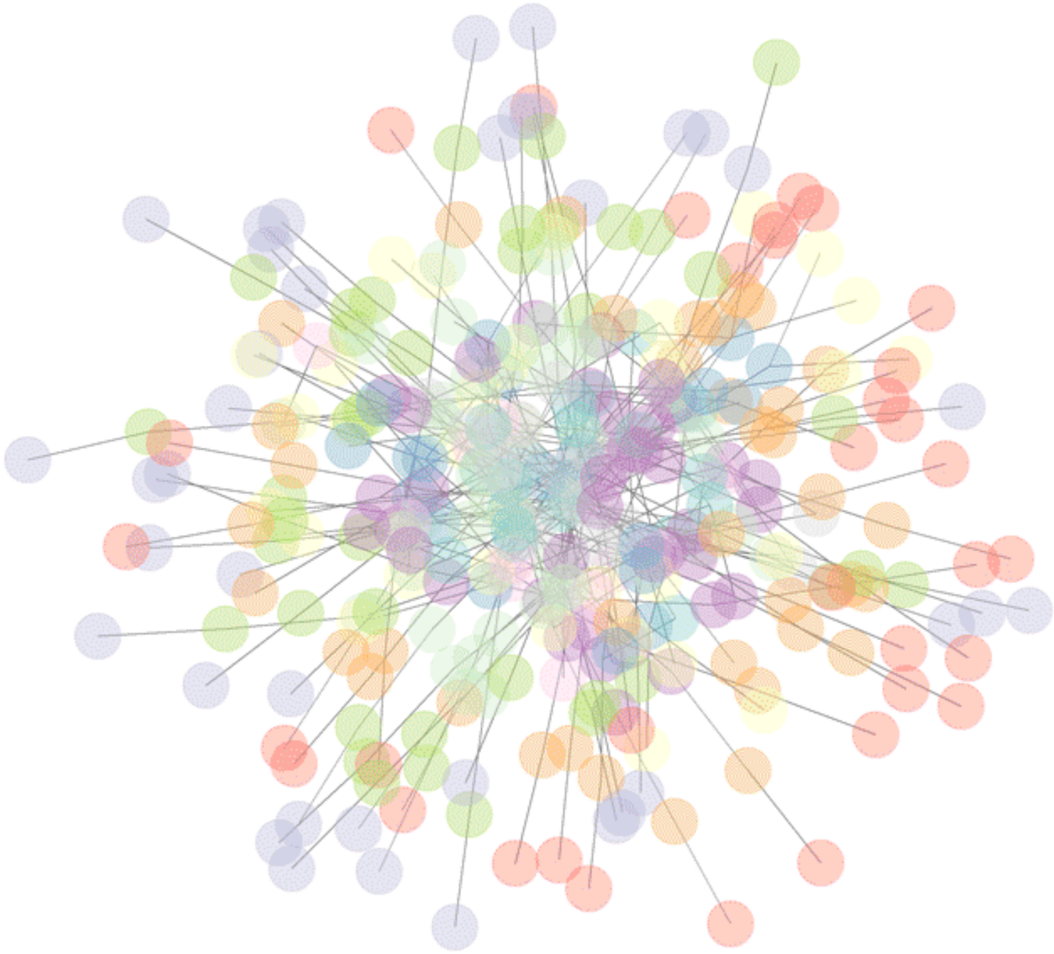
Results

Scenes

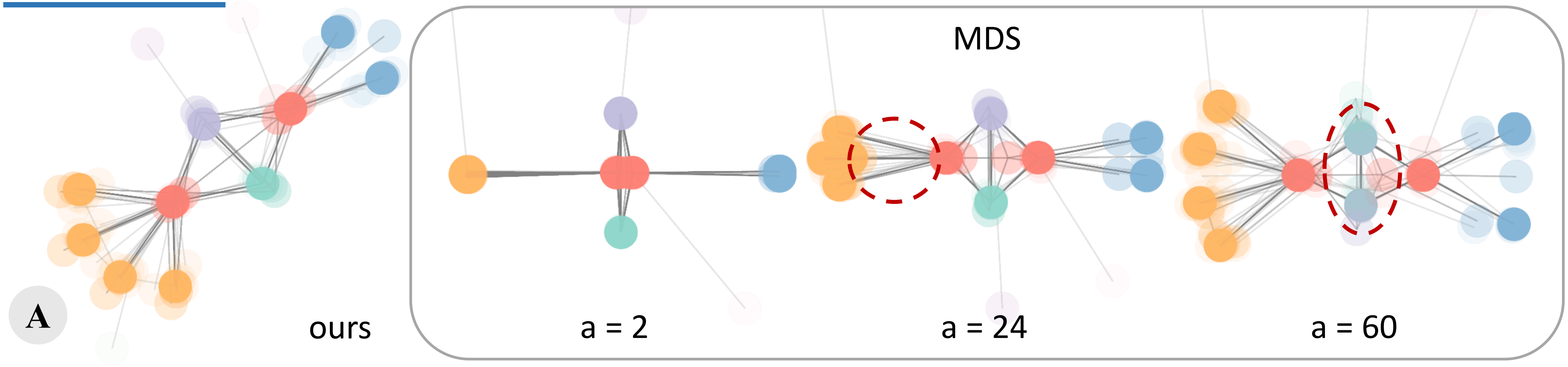


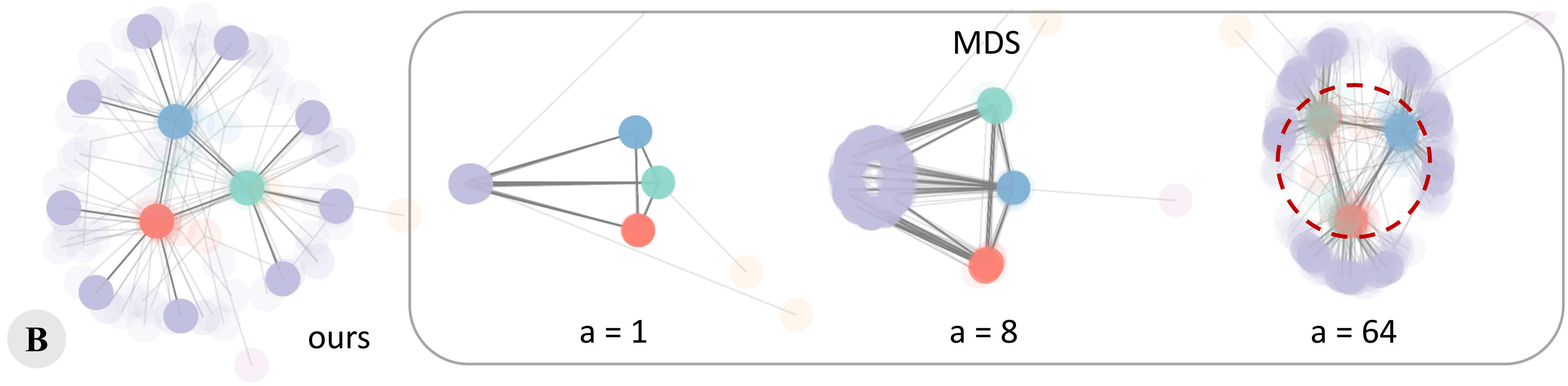
Results

Segmented meshes



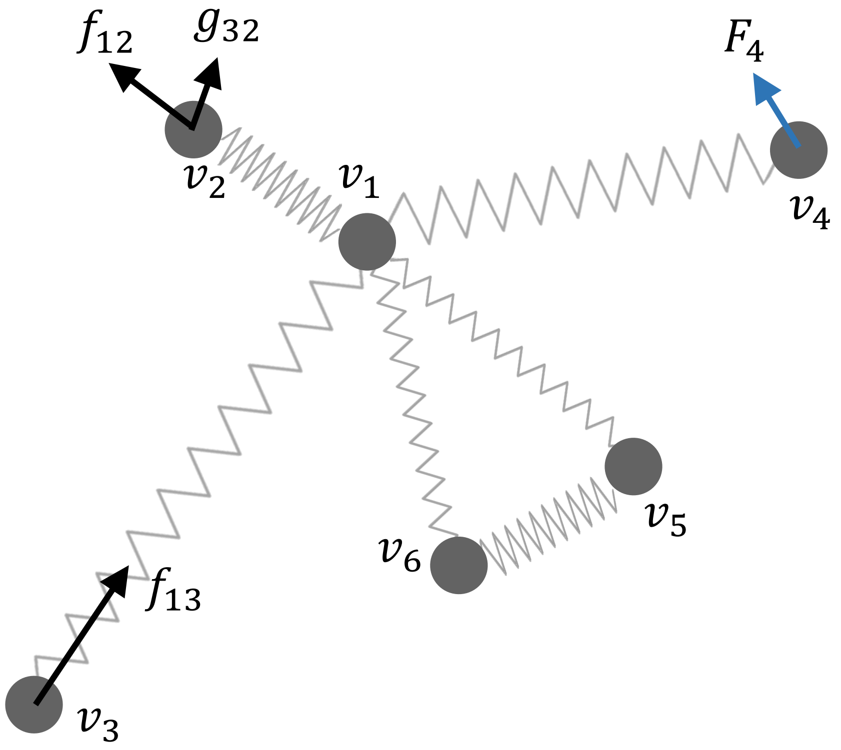
Results

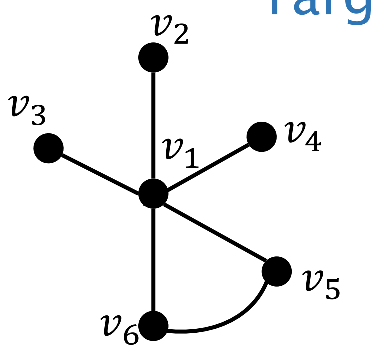




Graph Drawing

Force-directed method





Edges: springs – spring force f

Vertices: equally charged particles– electrical repulsion g

Graph Drawing

Spectral drawing method

Objective: the locations of the nodes that are connected to each other should be close.

$$E = \sum_{(v_i, v_j) \in E} w_{ij} \left\| x_i - x_j \right\|_2^2 = \mathrm{trace}(\, X^T L X \,)$$

L is the Laplacian matrix defined as $L = \mathrm{diag}(A\mathbf{1}) - A$

$$A = \begin{array}{c} \\ \\ \\ \\ \\ \\ \end{array}\begin{array}{cc} \begin{array}{cccccc} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \end{array} & \\ \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} & \begin{array}{c} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{array} \end{array}$$

Proposition

The minimizer of

$$\min_{X^T X = I_d} \mathrm{trace}(\, X^T L X \,)$$

is the eigenvectors of the Laplacian L corresponding to the first smallest d eigenvalues

$$L = \begin{array}{cc} \begin{array}{cccccc} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \end{array} & \\ \begin{pmatrix} 5 & -1 & -1 & -1 & -1 & -1 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 2 & -1 \\ -1 & 0 & 0 & 0 & -1 & 2 \end{pmatrix} & \begin{array}{c} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{array} \end{array}$$

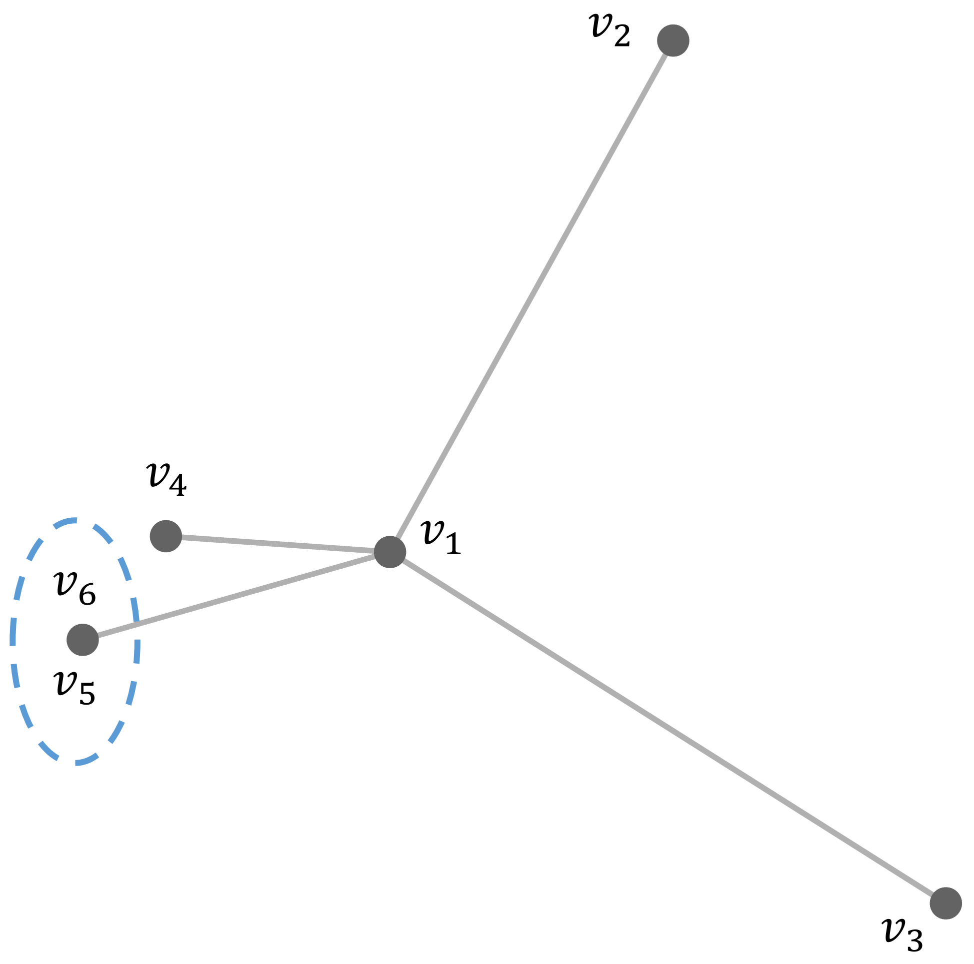
Note: by definition, L

1) is diagonally dominant $\Longrightarrow$ psd $\Longrightarrow$ all eigenvalues nonnegative
2) $L\mathbf{1} = 0\mathbf{1} \Longrightarrow 0$ is an eigenvalue w.r.t eigenvector $\frac{1}{\sqrt{n}}\mathbf{1}$

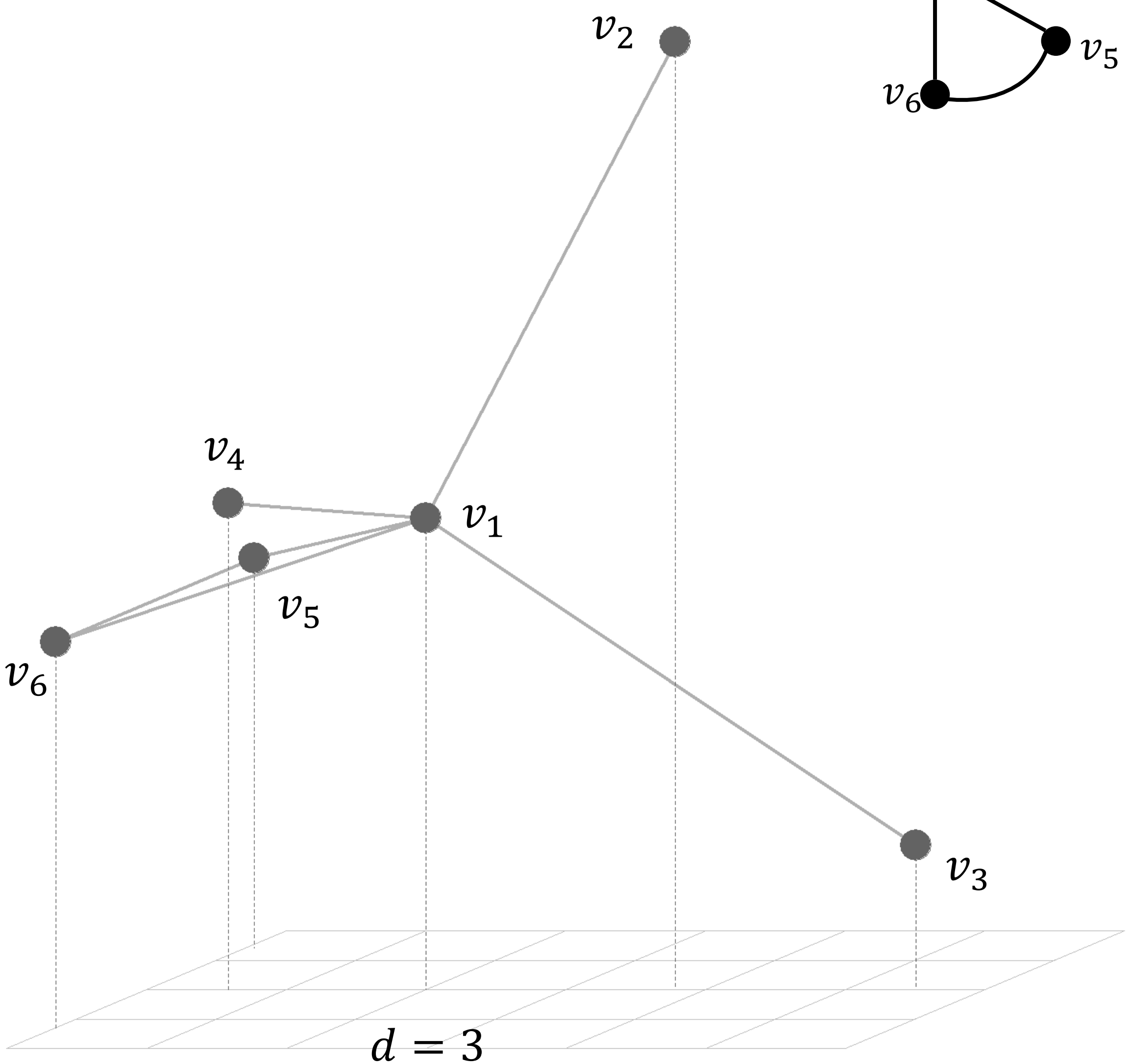
In general, we choose the eigenvectors w.r.t. nonzero eigenvalues

Graph Drawing

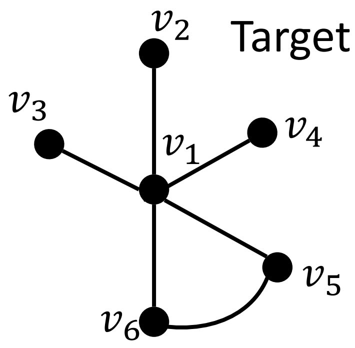
Spectral drawing method



$d = 2$



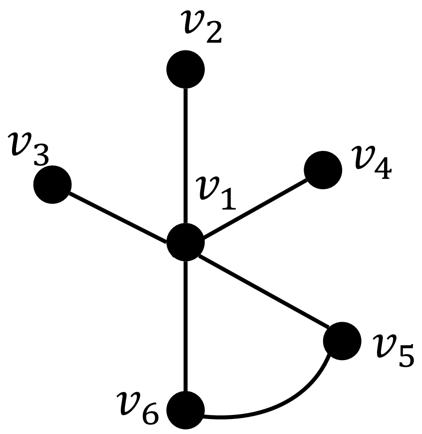
$d = 3$



Graph Drawing

Multidimensional Scaling (MDS)

Objective: the graph distance between a pair of nodes can be regarded as a dissimilarity measure, therefore, we could use MDS to find an embedding to preserve the dissimilarities.



Assume the graph distance d is given (can also be computed from matrix A), MDS tries to minimize:

$$E = \sum_{i=1}^{n}\sum_{j=1}^{n} w_{ij}\left(\|x_i - x_j\| - d_{ij}\right)^2$$

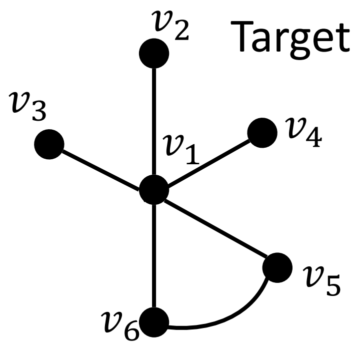
$$d = \begin{array}{cc} & \begin{array}{cccccc} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \end{array} \\ & \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 2 & 2 & 2 & 2 \\ 1 & 2 & 0 & 2 & 2 & 2 \\ 1 & 2 & 2 & 0 & 2 & 2 \\ 1 & 2 & 2 & 2 & 0 & 1 \\ 1 & 2 & 2 & 2 & 1 & 0 \end{pmatrix} \begin{array}{c} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{array} \end{array}$$

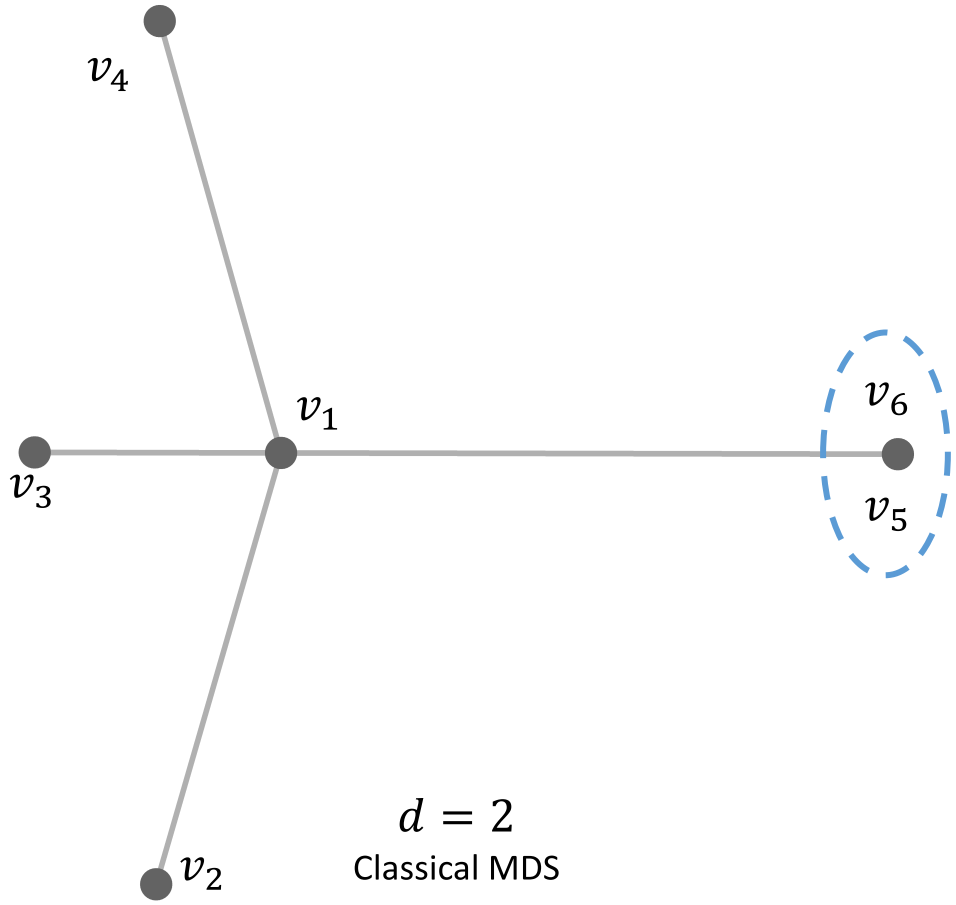
Non-convex problem – Stress Majorization method

- Convergence to a local minimum is guaranteed
- Easy to solve for each iteration

Graph Drawing

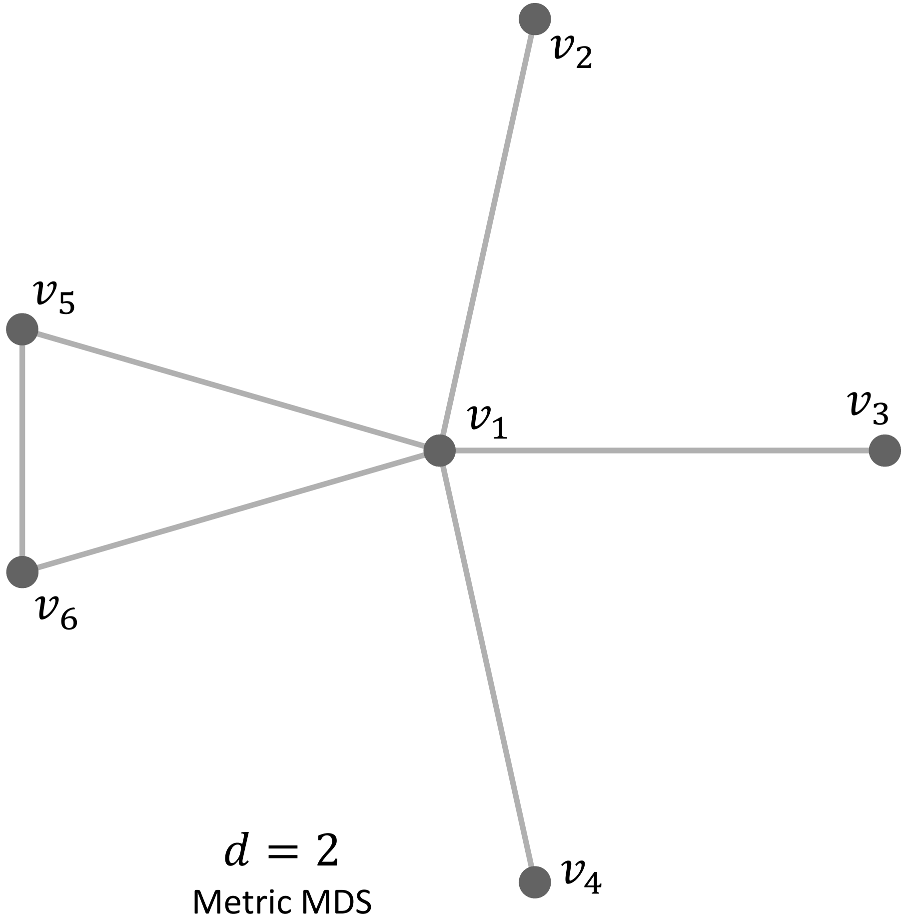
Multidimensional Scaling (MDS)





$d = 2$

Classical MDS



$d = 2$

Metric MDS

Tricks to construct majorizing function

Cauchy-Schwartz Inequality

The Cauchy Schwartz inequality:

$$\|x\|\|z\| \geq x^T z \Longrightarrow -\|x\| \leq -\frac{x^T z}{\|z\|}$$

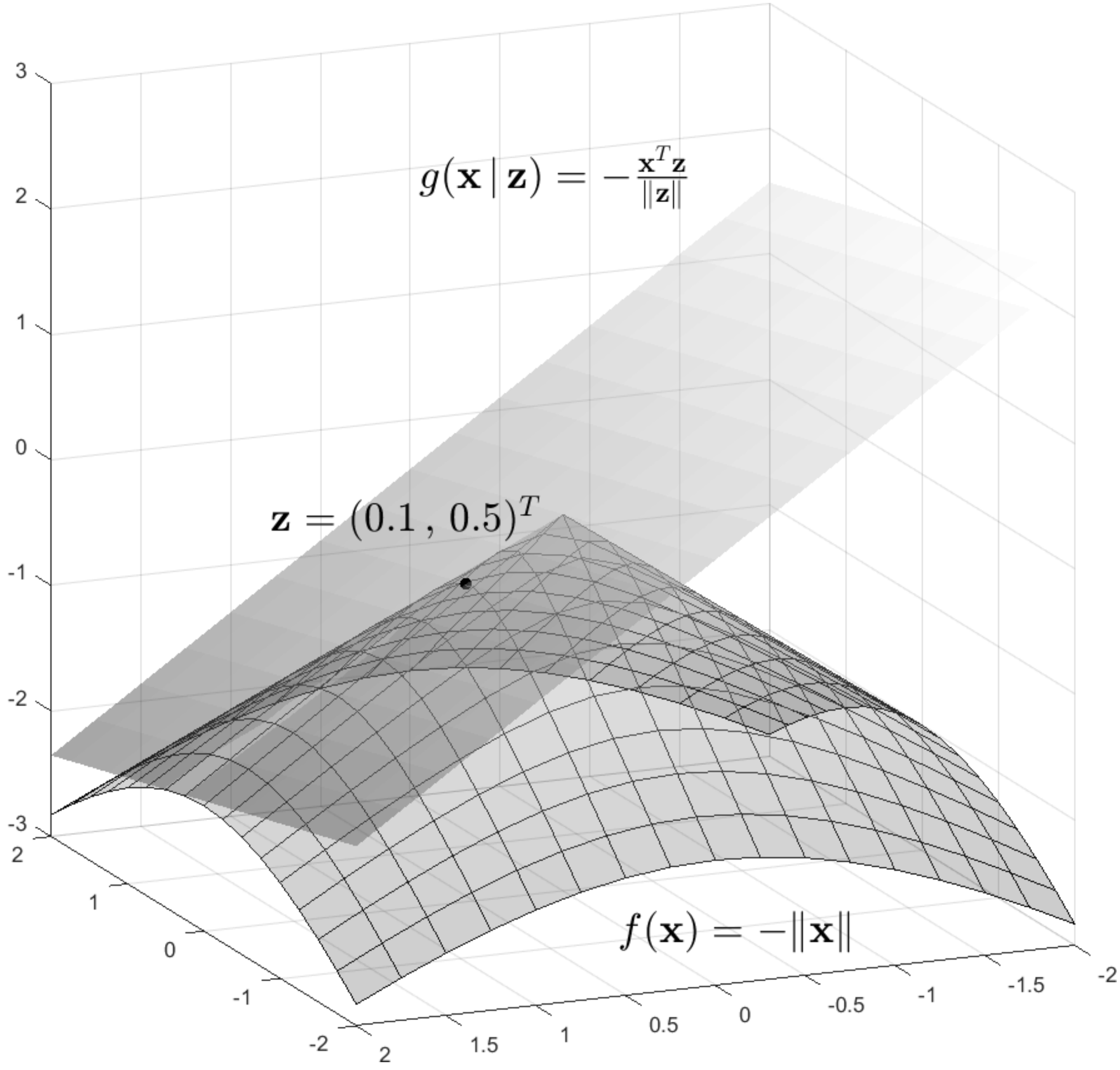
Denote $f(x) = -\|x\|, g(x|z) = -\frac{x^T z}{\|z\|}$

It's easy to check: $g(x|z) \geq f(x)$ and $g(z|z) = f(z)$

Recall the energy of the MDS

$$\sum_{i=1}^{n}\sum_{j=1}^{n} w_{ij}\left(\|x_i - x_j\| - d_{ij}\right)^2$$

It has terms $-2d_{ij}w_{ij}\|x_i - x_j\|$



Tricks to construct majorizing function

Via arithmetic-geometric mean Inequality

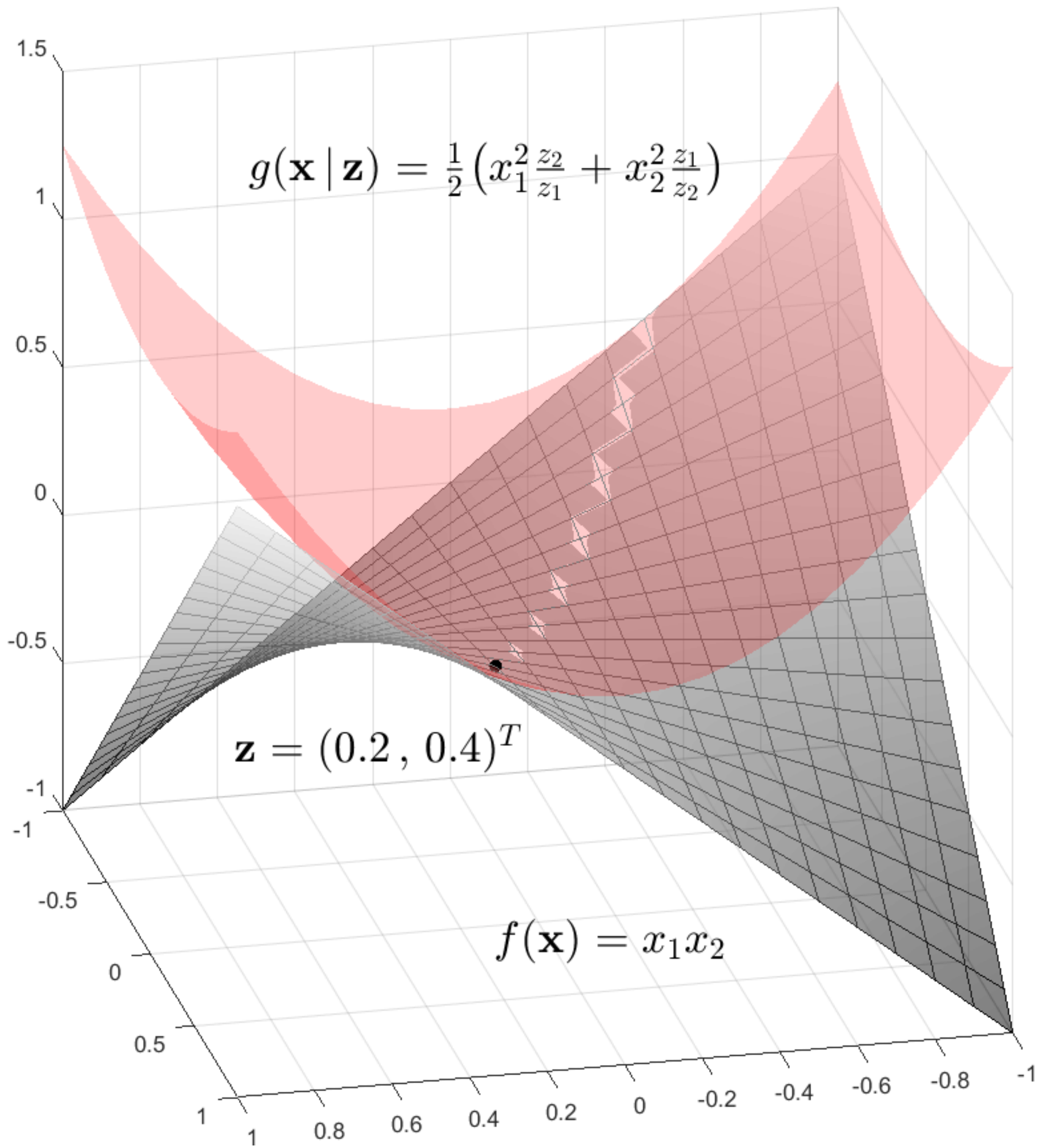
The arithmetic-geometric inequality:

$$\sqrt{ab} \leq \frac{a+b}{2} \Longrightarrow ab \leq \frac{a^2+b^2}{2}$$

Let $a = x_1\sqrt{\frac{z_2}{z_1}}, b = x_2\sqrt{\frac{z_1}{z_2}}$, we have

$$f(x_1, x_2) = x_1x_2 \leq \frac{1}{2}\left(x_1^2\frac{z_2}{z_1} + x_2^2\frac{z_1}{z_2}\right) \coloneqq g(x_1, x_2|z_1, z_2)$$

It's easy to check: $g(z|z) = f(z)$



Tricks to construct majorizing function

Via the definition of convexity

For a set of points $\{t_i\}_{i=1}^n$ and the sum-to-one weight $\{a_i\}_{i=1}^n$, a convex function $f(\cdot)$ satisfies:

$$f\left(\sum_{i=1}^n a_i t_i\right) \leq \sum_{i=1}^n a_i f(t_i)$$

Let $t_i = \frac{\theta_i(x_i - z_i)}{a_i} + \Theta^T z$, $a_i = \frac{\theta_i z_i}{\Theta^T z}$, we have

$$f(x) = f(\Theta^T x) \leq \sum_{i=1}^n \frac{\theta_i z_i}{\Theta^T z} f\left(\frac{x_i \Theta^T z}{z_i}\right) \coloneqq g(x|z)$$

It's easy to check: $g(z|z) = f(z)$